The Radiative Transfer Equation

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April 11, 2015

1 Objectives

- Derive the general RTE equation
- Derive the atmospheric 1D horizontally homogenous RTE equation
- Look at heating/cooling rates in some idealised examples

2 Some definitions

Let $I(x, \hat{s}, \nu)$ be the density of photons at $x$ propagating in direction $\hat{s}$ at frequency $\nu$. This is spectral irradiance or intensity. In time $dt$, this intensity will transport energy

$$\delta E = I(x, \hat{s}, \nu) dA_s d\Omega_s d\nu dt$$

(1)

Definitions:

- If $I(x, \hat{s}, \nu) = I(\hat{s}, \nu)$, the field is homogeneous
- If $I(x, \hat{s}, \nu) = I(x, \nu)$, the field is isotropic

Flux $F(x, \hat{d}, \nu)$ is defined as total power [J/s] flowing across a unit area perpendicular to $\hat{d}$ at frequency $\nu$. Different from $I$; in some applications confusing the two does not lead to problems. Projected area of $dA_d$ in the $\hat{s}$ direction:

$$dA_s = \cos(\hat{d}, \hat{s}) dA_d.$$  
(2)

We also must integrate over all possible $\hat{s}$ directions (equivalently, over solid angle $\Omega_s$), so

$$F(x, \hat{d}, \nu) dA_d d\nu = \int_{\Omega_s} I(x, \hat{s}, \nu) dA_s d\Omega_s d\nu$$

(3)

$$F(x, \hat{d}, \nu) = \int_{\Omega_s} I(x, \hat{s}, \nu) \cos(\hat{d}, \hat{s}) d\Omega_s.$$  
(4)
3 Radiative Transfer Equation (General)

From now on we are monochromatic (single frequency), until I state otherwise. Use $I, F$ for monochromatic quantities, $i, f$ for $\nu$-averaged quantities.

Start with
\[
\frac{dI}{ds} = -kI
\]
(5)
\[
\frac{dI}{ds} = -\kappa\rho I
\]
(6)

$k$ is $m^{-1}$, $\kappa$ is mass extinction cross-section in $m^2/kg$. Extinction just means “removal from the beam”: absorption or scattering.

Mass extinction cross-section depends strongly on frequency. Also on pressure and temperature. Also more obviously must depend on amount of absorbers. Generally we can write:
\[
\kappa(\nu, p, T) = \sum_{i=0}^{n} \kappa_i(\nu, p, T)q_i(p)
\]
(7)

with $q$ the specific concentration of a species $[kg/kg]$ and $i$ a sum over all species.

Add in (arbitrary) source term $J$ to get most general (monochromatic) equation of radiative transfer possible
\[
1\frac{dI}{\kappa\rho ds} = -I + J.
\]
(8)

Each quantity is defined at a single frequency $\nu$ only and $s$ is the path. Sign convention: $J$ is $+ve$ for source. Can be expanded using chain rule to incorporate arbitrary number of beams as
\[
-\kappa\frac{s \cdot \nabla I(x, s)}{\kappa\rho} = I(x, s) - J(x, s)
\]
(9)

where $x$ is position and $s$ is the directional unit vector. Note 6 degrees of freedom at each point, not 3.

In Cartesian coordinates
\[
\frac{d}{ds} = s \cdot \nabla = \hat{s}_x \partial_x + \hat{s}_y \partial_y + \hat{s}_z \partial_z.
\]
(10)

Often we be dealing with situations where quantities only vary with height. Plane parallel assumption:
\[
-\mu \frac{\kappa \rho}{dz} \frac{dI(z, \mu)}{dz} = I(z, \mu) - J(z, \mu)
\]
(11)

where $\mu$ is $\cos \theta$: the angle made between the direction of propagation and the vertical.

Define optical depth:
\[
d\tau = \int_{z_0}^{z} \kappa(z) \rho(z) dz.
\]
(12)

It’s dimensionless. Use hydrostatic equation and integrate to get
\[
\int_{\tau}^{\infty} d\tau' = -\frac{1}{g} \int_{p_0}^{p} \kappa dp'.
\]
(13)

For constant $\kappa \neq \kappa(z)$,
\[
\tau - \tau_{\infty} = \frac{-p_{\infty}}{g}
\]
(14)
But when $\tau = 0$, $p = p_s$, so
\[
\tau_\infty = \frac{\kappa}{g} p_s = \kappa u
\]
(15)
where $u$ is atmospheric mass column (or path if you like). Note path can be defined more generally for a minor species between two pressure levels as
\[
u(p_1, p_2) = - \int_{p_1}^{p_2} q(p) \frac{dp}{g}
\]
(16)
where $q(p)$ is the specific concentration $[\text{kg/kg}]$.

Using this convention, which I’ll try to stick to for most of course,
\[
\mu d\mathcal{I}(\tau, \mu) \frac{d\tau}{d\mu} = -\mathcal{I}(\tau, \mu) + \mathcal{J}(\tau, \mu)
\]
(17)
with $\mu = \cos \theta$ and $\theta$ the angle in a single plane. This is as general an equation as we’ll ever need for climate. When we discuss retrievals later in course, we may need to bring back $\phi$.

4 Source terms and phase functions

In longwave, if we can assume LTE, $\mathcal{J}_{\nu} = B_{\nu}[T(\tau)]$, the Planck source function. Then we have Schwarzschild equation (1st derived for stars)
\[
\mu \frac{d\mathcal{I}(\tau, \mu)}{d\tau} = -\mathcal{I}(\tau, \mu) + B_{\nu}(\tau).
\]
(18)
In visible, things are more complicated. First thing to take into account is the stellar source term.

This can be taken into account by thinking about propagating solar beam. How much gets to a given layer? If incoming flux is $F_{\odot}$, result is $F_{\odot} e^{(\tau - \tau_\infty)} / \mu_0$, given stellar zenith angle $\theta_0$. Note our equation is for the diffuse intensity only.

Now how much stellar light is scattered? Well, total extinction cross-section $\kappa = \kappa_a + \kappa_s$. We can define single scattering albedo
\[
\tilde{\omega} \equiv \frac{\kappa_s}{\kappa_e}
\]
(19)
We must also define the phase function $\mathcal{P}(\mu, -\mu_0)$, which tells us how EM radiation is redistributed as a function of angle. Note for Rayleigh scattering
\[
\mathcal{P}(\Theta) = \frac{3}{4} (1 + \cos^2 \Theta).
\]
(20)
Putting things together gives
\[
\mathcal{J}_{\text{stel}} = \tilde{\omega} \frac{\mathcal{P}(\mu, -\mu_0)}{4\pi} F_{\odot} e^{(\tau - \tau_\infty)} / \mu_0
\]
(21)
Inclusion of $\tilde{\omega}$ makes sure we are only including photons that get scattered (not absorbed). Remember $\tau$ is the extinction optical depth.

$4\pi$ is normalization factor (solid angle of sphere). Full phase function gives
\[
\int_{0}^{2\pi} \int_{-1}^{1} \mathcal{P}(\mu, \phi; \mu', \phi') d\mu' d\phi' = 4\pi
\]
(22)
if azimuth is included. But we’re working in azimuth-independent limit here.

We’re not finished: we also need to account for the diffuse (multiple scattering) source. This can be got by integrating $I$ over $\mu$ times the phase function:

$$J_{\text{diff}} = \tilde{\omega} \frac{4\pi}{2} \int_{-1}^{1} \int_{0}^{2\pi} I(\tau, \mu', \phi') P(\mu, \phi; \mu', \phi') d\mu' d\phi'$$ (23)

$$J_{\text{diff}} = \tilde{\omega} \frac{2}{2} \int_{-1}^{1} I(\tau, \mu') P(\mu, \mu') d\mu'$$ (24)

Hence altogether

$$\mu \frac{dI(\tau, \mu)}{d\tau} = -I(\tau, \mu) + J_{\text{stel}} + J_{\text{diff}}$$ (25)

$$\mu \frac{dI(\tau, \mu)}{d\tau} = -I(\tau, \mu) + \tilde{\omega} \frac{4\pi}{2} P(\mu, -\mu_0) F_{\odot} e^{(\tau - \tau_{\infty})/\mu_0} + \tilde{\omega} \frac{2}{2} \int_{-1}^{1} I(\tau, \mu') P(\mu, \mu') d\mu'.$$ (26)

This is progress, but still an insanely complex integrodifferential equation. To get analytical results or even do a numerical simulation, we need to make further simplifications.

Most fundamental is to discretize in $\mu$ and replace integral with a sum. Then we can write

$$\mu_i \frac{dI(\tau, \mu_i)}{d\tau} = -I(\tau, \mu_i) + \tilde{\omega} \frac{4\pi}{2} P(\mu_i, -\mu_0) F_{\odot} e^{(\tau - \tau_{\infty})/\mu_0} + \tilde{\omega} \frac{2}{2} \sum_{j=-n}^{n} I(\tau, \mu_j) P(\mu_i, \mu_j) a_j.$$ (27)

### 4.1 Legendre Polynomial Expansion

One of the most common ways to do the expansion is with Legendre polynomials. Legendre polynomials are the solutions to the 2nd order ODE

$$\frac{d}{d\mu} \left[ (1 - \mu^2) \frac{dy}{d\mu} \right] + n(n + 1)y = 0$$ (28)

that appears when you solve Laplace’s equation

$$\nabla^2 \Phi = 0$$ (29)

in spherical coordinates. Power series method is used to solve Legendre’s equation (we won’t cover that here). Legendre polynomials are defined as

$$L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[ (x^2 - 1)^n \right]$$ (30)

They are orthogonal on the interval $[-1, 1]$

$$\int_{-1}^{1} L_m(x) L_n(x) dx = \frac{2}{2n + 1} \delta_{mn}$$ (31)

Now assume $P(\mu, \mu') = P(\gamma)$ is an arbitrary function on $-1 < \mu < 1$ (here $\gamma$ is the cosine of the scattering angle $\gamma = \cos \Theta$). So we can write

$$P(\gamma) = \sum_{l=0}^{\infty} p_l L_l(\gamma)$$ (32)
If we were working in full 3D, addition theorem would give us

\[ \mathcal{P}(\gamma) = \sum_{m=0}^{\infty} \sum_{l=m}^{\infty} p_l^m L_l^m(\mu) L_l^m(\mu') \cos(m \phi' - \phi) \]  

(33)

with

\[ p_l^m = (2 - \delta_{0,m}) p_l (l - m)! \frac{(l + m)!}{m!} \]  

(34)

The \( L_l^m \) are the associated Legendre polynomials.

For the \( \phi \)-independent case, we can just write \( m = 0 \), though, giving

\[ p_l^m = p_l \]  

(35)

and

\[ \mathcal{P}(\gamma) = \sum_{l=0}^{\infty} p_l L_l(\mu) L_l(\mu') \]  

(36)

We can hence also write the scattering term as

\[ \frac{\tilde{\omega}}{2} \sum_{j=-n}^{n} \sum_{l=0}^{\infty} I(\tau, \mu_j) p_l L_l(\mu_i) L_l(\mu_j) a_j. \]  

(37)

Finally, note that from spherical geometry the scattering angle \( \Theta \) can be written as

\[ \cos \Theta = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi') \]  

(38)

or

\[ \gamma = \mu \mu' + \sqrt{1 - \mu^2} \sqrt{1 - \mu'^2} \cos(\phi - \phi'). \]  

(39)

5 Two-stream equation (longwave)

Fortunately life is a bit easier when you can neglect scattering. We start from the previously derived

\[ \mu \frac{dI(\tau, \mu)}{d\tau} = -I(\tau, \mu) + B_\nu(\tau) \]  

(40)

To proceed we make use of the fact that for IR, the source function is isotropic (as we’ve noted already).

Assume that same goes for \( I \). Then we can integrate over solid angle in 1 hemisphere only. This gives us a flux quantity \([W/m^2/Hz]\]

\[ F_+ = \int_{\Omega_+} I \cos \theta d\Omega \]  

(41)

Note the \( \cos \theta \) term — we are taking the normal component of \( I \) to the vertical.

\[ F_+ = \int_0^{2\pi} \int_0^{\pi/2} I \cos \theta \sin \theta d\theta d\phi. \]  

(42)
Given isotropy, we then have
\[ F_+ = +\pi I. \] (43)

C.f. \( \sigma T^4 = \pi \int B_\nu d\nu \). Similarly,
\[ F_- = \int_0^{2\pi} \int_0^0 I \cos\theta \sin\theta d\theta d\phi \] (44)
\[ F_- = -\pi I. \] (45)

Now integrate the whole RTE equation over solid angle \( \int_{\Omega_i} d\Omega \) to get
\[ \frac{d}{d\tau} \int_{\Omega_i} I \cos\theta d\Omega = -2\pi I + 2\pi B_\nu \] (46)
\[ \frac{1}{2} \frac{dF_+}{d\tau} = -F_+ + \pi B_\nu \] (47)

Looking back at eqn. at start of section with \( \mu \), we can identify \( \mu = \cos\theta = \frac{1}{2} \). Other values are possible depending on how we integrated. This is sometimes known as the diffuse approximation. Now we fold this into \( \tau \), such that \( \tau/\mu \to \tau \). We must always be clear on which version of \( \tau \) we’re using!

Finally,
\[ \frac{dF_+}{d\tau} = -F_+ + \pi B_\nu \] (48)
\[ \frac{dF_-}{d\tau} = +F_- - \pi B_\nu \] (49)

These are our IR 2-stream equations. Signs in \( F_- \) equation comes from definition \( F_- = -\pi I \).

6 Solutions to the 2-stream equations

We’ve neglected scattering, which makes these equations uncoupled. Linear inhomogeneous 1st order ODEs in fact. General solution: try \( F_+ = A(\tau) e^{-\tau} \), which gives for upward branch
\[ \frac{dF_+}{d\tau} = e^{-\tau} \frac{dA}{d\tau} - F_+ = -F_+ + \pi B_\nu \] (50)
so
\[ \frac{dA}{d\tau} = \pi B_\nu e^{+\tau} \] (51)

hence
\[ \int_0^\tau d\tau' F_+(\tau + \tau') = \pi \int_0^\tau B_\nu(\tau') e^{+\tau'} d\tau' \] (52)
\[ F_+(\tau) = F_+(0) e^{-\tau} + \pi \int_0^\tau B_\nu(\tau') e^{\tau' - \tau} d\tau' \] (53)

and for \( F_- = C(\tau) e^{+\tau} \), yielding
\[ F_-(\tau) = F_-(0) e^{-\tau} + \pi \int_\tau^{\tau_\infty} B_\nu(\tau') e^{\tau' - \tau} d\tau'. \] (54)
Heating rates come from thinking about how energy gets deposited in a layer. Per unit optical depth, the relevant quantity is

\[ Q = -\frac{dF}{d\tau} = -\frac{d(F_+ - F_-)}{d\tau} \quad (55) \]

Sign is necessary because of how we’ve defined \( \tau \) (increasing with height). We often want heating rate per unit mass [W/kg] though. To get this, remember \( dp = -gdm \). Hence

\[ \dot{Q} \equiv g\frac{dF}{dp} = -g\frac{d\tau}{dp}Q \quad (56) \]

Because we’re using the 2-stream optical depth (includes \( \cos\theta \)),

\[ \frac{d\tau}{dp} = -\frac{\kappa}{g\cos\theta} \quad (57) \]

So

\[ \dot{Q} \equiv \frac{\kappa}{\cos\theta}Q. \quad (58) \]

Now for some solutions.

**Beer’s Law**

Simplest case possible: atmosphere has no source function (e.g. too cold at given frequency). Then \( B_\nu = 0 \) and \( F_+ = F_+(0)e^{-\tau} \) etc.

**Infinite isothermal medium**

Consider a solution \( F_+ = F_- = \pi B_\nu \). Then

\[ \pi B_\nu = \pi B_\nu e^{-\tau} + \pi B_\nu \int_0^\tau e^{-(\tau - \tau')} d\tau' \]

\[ 1 - e^{-\tau} = 1 - e^{-\tau} \quad (59) \]

So it’s a solution. No heating either, consistent with what we already know about isothermal radiation.

**Finite thickness isothermal slab**

Define \( \tau = 0 \) at the centre, \( \tau = \pm \frac{1}{2}\tau_b \) (or \( \tau_\infty \)) at the top and bottom. No incident flux, so \( F_+(-\tau_b/2) = F_-(\tau_b/2) = 0 \). Furthermore, because layer is isothermal we can do the exponential easily:

\[ F_+ = \pi B_\nu \int_{-\frac{1}{2}\tau_b}^\tau e^{\tau'-\tau} d\tau' \quad (61) \]

\[ F_- = \pi B_\nu \left[ 1 - e^{-\left(\tau + \frac{1}{2}\tau_b\right)} \right] \quad (62) \]

Downward version is

\[ F_- = -\pi B_\nu \left[ 1 - e^{-\left(\tau - \frac{1}{2}\tau_b\right)} \right] \quad (63) \]
Heating rate

\[ Q = -\frac{d(F_+ - F_-)}{d\tau} \]  \hspace{1cm} (64)

\[ Q = -\pi B_\nu e^{-\frac{1}{2} \tau_b} \left[ e^{-\tau} + e^\tau \right] \]  \hspace{1cm} (65)

In optically thin limit we have constant cooling. In thick limit, boundary layer cooling. Condensed water droplets absorb very effectively (more on this later). So thick limit can apply for low-lying (stratus) clouds.