AM 106/206: Applied Algebra **Prof. Salil Vadhan**

Problem Set 0

Assigned: Tue. Sept. 7, 2010 Due: Fri. Sept. 10, 2010 (2:10 pm)

- This problem set is optional and will not count for your grade. However, if you have not taken a prior proof-based math course, it is strongly encouraged that you complete and turn in the problem set for practice and feedback on doing proofs.
- You may submit your problem sets in the AM106 in the Maxwell–Dworkin basement, or electronically by email to `am106-hw@seas.harvard.edu`. If you use LaTeX, please submit both the source (`.tex`) and the compiled file (`.pdf`). Name your files `PS0-yourlastname`.

Problem 1. (Proof by Contradiction) Joe the painter has 2010 cans of paint. Show that at least one of the following statements is true about Joe's paint collection.

- Among the cans, there are at least 42 of them with the same color.
- Among the cans, there are at least 50 different colors of paint.

Problem 2. (Set Equality) Which of the following is true? Prove your answers.

- For every three sets A, B, C, we have $A \cup (B \cap C) = (A \cup B) \cap C$.
- For every three sets A, B, C, we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Problem 3. (Induction) The Fibonacci numbers $F_0, F_1, \ldots$ are defined inductively by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$. Thus the sequence (starting at F_0) is $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$. Prove by induction that for $n \geq 2$, $F_n \geq \varphi^{n-2}$, where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio.

Problem 4. (Incorrect Induction) What is the wrong with the following proof by induction?

Claim: In every set of n students, all students have the same height.
"Proof" by Induction:

- Base Case: For every set of size 1, the claim is clearly true (all the students in that set have the same height).
- Induction Step: Assume that the claim is true for sets of k students (this is the induction hypothesis), and we'll prove that it also holds for sets of $k + 1$ students. Consider an arbitrary set S consisting of $k + 1$ students, say $S = \{p_1, \ldots, p_{k+1}\}$. Let $S' = \{p_1, \ldots, p_k\}$. Since $|S'| = k$, our induction hypothesis tells us that all students in S' have the same height. So now we only need to show that p_{k+1} has the same height too. To do this, consider the set $S'' = \{p_2, \ldots, p_{k+1}\}$. Since $|S''| = k$, the induction hypothesis also tells us that all students in S'' have the same height. In particular, p_{k+1} has the same height as p_2, and hence the same height as all students in S'.