AM 106/206: Applied Algebra **Prof. Salil Vadhan**

Problem Set 6

Assigned: Fri. Oct. 22, 2010 Due: Fri. Oct. 29, 2010 (2:10 PM sharp)

- You may submit your problem sets in the AM106 in the Maxwell–Dworkin basement, or electronically by email to `am106-hw@seas.harvard.edu`. If you use LaTeX, please submit both the source (`.tex`) and the compiled file (`.pdf`). Name your files `PS6-yourlastname`.
- Aim for clarity and conciseness in your solutions, emphasizing the main ideas over low-level details. Justify your answers except when otherwise specified.
- Problems marked [AM106] or [AM106-X] are for AM106 students (though AM206 students should confirm that they know how to do them), and those marked [AM206-X] are for AM206 students. However, AM106 students can do a problem marked [AM206-X] instead of one marked [AM106-X] (for the same value of X) if they wish (out of interest, or for a challenge). If you wish to keep the option of staying in either AM106 or AM206 open until add/drop date, then you should do all problems marked [AM106] and all problems marked [AM206-X].

Problem 1. (Homomorphisms) Determine all homomorphisms from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{84}$. For each, identify the kernel and image. (Hint: how does $\varphi(x)$ relate to $\varphi(1)$? And what can we say about the order of $\varphi(1)$?)

Problem 2. (Cautions with Normality and Factor Groups)

1. Give an example of a group G and subgroups N and H such that $H \lhd N \lhd G$, but $H \ntriangleleft G$.
2. Give an example of a group G and normal subgroup N such that $G/N \times N \ncong G$.

Justify your answers.

Problem 3. (Normal Subgroups of Small Index)

1. [AM106-A] Show that if H is a subgroup of G of index 2, then H is normal in G.
2. [AM206-A] Show that if G has no subgroups of index 2, then every subgroup of index 3 is normal. (Hint: Multiplication on the left by $g \in G$ permutes the cosets of H. This gives rise to a homomorphism $\varphi : G \to S_3$. Reason about $\mathrm{Ker}(\varphi)$, $\mathrm{Im}(\varphi)$, and $\varphi^{-1}(A_3)$.)

Problem 4. (Factor Groups and Homomorphisms) For each of the following groups G and subsets $H \subseteq G$, determine whether H is a normal subgroup of G. If yes, then find a familiar group G' such that $G/H \cong G'$. Prove that $G/H \cong G'$ by giving an appropriate homomorphism from G to G'.

1. $G = \mathbb{Z}$, $H = \{\text{prime integers}\}$.
2. $G = \mathbb{Z} \times \mathbb{Z}$, $H = \{(n, n) : n \in \mathbb{Z}\}$.
3. $G = S_5 \times S_5$, $H = \{(\sigma, \sigma) : \sigma \in S_5\}$.
4. $G = \mathbb{C}^*$, $H = S^1 = \{z \in \mathbb{C}^* : |z| = 1\}$.
5. $G = \{\text{affine linear functions } x \mapsto ax + b \text{ from } \mathbb{R} \text{ to } \mathbb{R}, \text{ with } a \in \mathbb{R}^*, b \in \mathbb{R}\}$ (under composition), $H = \{\text{linear functions } x \mapsto ax, \text{ with } a \in \mathbb{R}^*\}$.
6. [AM106-B] $G = \mathbb{Z}_{11}^*$, $H = \{\text{the squares in } G\}$.
7. [AM206-B] $G = \mathbb{Z}_{77}^*$, $H = \{\text{the squares in } G\}$.

Problem 5. (Rings) Which of the following are rings? integral domains? fields? Justify your answers.

1. $\mathbb{N}$
2. $\mathbb{Z}_5[x]$
3. $\mathbb{Z}_9[x]$
4. $GL_n(\mathbb{R})$
5. $\mathbb{Z}_3 \times \mathbb{Z}_7$