Problem 1. (Homomorphisms) Determine all homomorphisms from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{84}$. For each, identify the kernel and image. (Hint: how does $\varphi(x)$ relate to $\varphi(1)$? And what can we say about the order of $\varphi(1)$?)

Problem 2. (Cautions with Normality and Factor Groups)

1. Give an example of a group $G$ and subgroups $N$ and $H$ such that $H \triangleleft N \triangleleft G$, but $H \not\triangleleft G$.

2. Give an example of a group $G$ and normal subgroup $N$ such that $G/N \times N \not\cong G$.

Justify your answers.

Problem 3. (Normal Subgroups of Small Index)

1. [AM106-A] Show that if $H$ is a subgroup of $G$ of index 2, then $H$ is normal in $G$.

2. [AM206-A] Show that if $G$ has no subgroups of index 2, then every subgroup of index 3 is normal. (Hint: Multiplication on the left by $g \in G$ permutes the cosets of $H$. This gives rise to a homomorphism $\varphi : G \to S_3$. Reason about $\text{Ker}(\varphi)$, $\text{Im}(\varphi)$, and $\varphi^{-1}(A_3)$.)
Problem 4. (Factor Groups and Homomorphisms) For each of the following groups $G$ and subsets $H \subseteq G$, determine whether $H$ is a normal subgroup of $G$. If yes, then find a familiar group $G'$ such that $G/H \cong G'$. Prove that $G/H \cong G'$ by giving an appropriate homomorphism from $G$ to $G'$.

1. $G = \mathbb{Z}$, $H = \{\text{prime integers}\}$.
2. $G = \mathbb{Z} \times \mathbb{Z}$, $H = \{(n, n) : n \in \mathbb{Z}\}$.
3. $G = S_5 \times S_5$, $H = \{(\sigma, \sigma) : \sigma \in S_5\}$.
4. $G = \mathbb{C}^*$, $H = S^1 = \{z \in \mathbb{C}^* : |z| = 1\}$.
5. $G = \{\text{affine linear functions } x \mapsto ax + b \text{ from } \mathbb{R} \text{ to } \mathbb{R}, \text{ with } a \in \mathbb{R}^*, b \in \mathbb{R}\}$ (under composition), $H = \{\text{linear functions } x \mapsto ax, \text{ with } a \in \mathbb{R}^*\}$.
6. [AM106-B] $G = \mathbb{Z}_{11}^*$, $H = \{\text{the squares in } G\}$.
7. [AM206-B] $G = \mathbb{Z}_7^*$, $H = \{\text{the squares in } G\}$.

Problem 5. (Rings) Which of the following are rings? integral domains? fields? Justify your answers.

1. $\mathbb{N}$
2. $\mathbb{Z}_5[x]$
3. $\mathbb{Z}_9[x]$
4. $GL_n(\mathbb{R})$
5. $\mathbb{Z}_3 \times \mathbb{Z}_7$