AM 106/206: Applied Algebra	**Prof. Salil Vadhan**

Problem Set 7

Assigned: Fri. Oct. 29, 2010 — Due: Fri. Nov. 5, 2010 (2:10 PM sharp)

- You may submit your problem sets in the AM106 in the Maxwell–Dworkin basement, or electronically by email to `am106-hw@seas.harvard.edu`. If you use LaTeX, please submit both the source (`.tex`) and the compiled file (`.pdf`). Name your files `PS7-yourlastname`.
- Aim for clarity and conciseness in your solutions, emphasizing the main ideas over low-level details. Justify your answers except when otherwise specified.
- Problems marked [AM106] or [AM106-X] are for AM106 students (though AM206 students should confirm that they know how to do them), and those marked [AM206-X] are for AM206 students. However, AM106 students can do a problem marked [AM206-X] instead of one marked [AM106-X] (for the same value of X) if they wish (out of interest, or for a challenge). If you wish to keep the option of staying in either AM106 or AM206 open until add/drop date, then you should do all problems marked [AM106] and all problems marked [AM206-X].

Problem 1. (The Si(111) Reconstructed Face) Attached is a piece of the reconstructed Si(111) face, which is repeated infinitely to form a 2-D crystal F. (This face is obtained by cutting a 3-D silicon crystal along a different plane than the one giving the Si(100) face seen in lecture.)

1. On the attached diagram, draw two vectors that generate the translation lattice of F.
2. Find and mark a point p of maximal rotational symmetry, and determine the group $\mathrm{Point}(F, p)$.
3. Use the flowchart in Gallian Figure 28.18 to classify $\mathrm{Isom}(F, p)$ among the 17 2-D crystallographic groups.
4. Using generators for $\mathrm{Point}(F, p)$, determine whether the diffusivity of the Si(111) face is isotropic.

Problem 2. (From Translations and Point Groups to the Full Symmetry Group) Let E_2 be the 2-dimensional Euclidean group, and $F : X \to \mathbb{R}^2$ be a 2-dimensional crystal.

1. Let E_2^+ denote the set of rotations in E_2, i.e. the set of isometries of the form $T(x) = \mathrm{Rot}_\theta x + b$, for $\theta \in [0, 2\pi)$ and $b \in \mathbb{R}^2$. Show that E_2^+ is a subgroup of E_2, and that it is of index 2.
2. Let $\mathrm{Isom}(F)^+ = \mathrm{Isom}(F) \cap E_2^+$. Show that either $\mathrm{Isom}(F)^+ = \mathrm{Isom}(F)$ or $\mathrm{Isom}(F)^+$ is a subgroup of $\mathrm{Isom}(F)$ and that it is of index 2. Similarly, for a point $p \in \mathbb{R}^2$, if we define $\mathrm{Point}(F, p)^+ = \mathrm{Point}(F, p) \cap E_2^+$ then $\mathrm{Point}(F, p)^+$ either equals $\mathrm{Point}(F, p)$ or is a subgroup

of $\mathrm{Point}(F,p)$ of index 2. (Hint: these statements are have nothing to do with geometry, and generalize to studying the intersection H^+ of arbitrary subgroups G^+, H of a group G such that $[G : G^+] = 2$.)

3. Let $\mathrm{Rot}(F) = \{\mathrm{Rot}_\theta : \exists b \text{ s.t. } T(x) = \mathrm{Rot}_\theta x + b \text{ is in } \mathrm{Isom}(F)\}$. Show that $\mathrm{Rot}(F)$ is a cyclic group generated by Rot_{θ^*} for the smallest positive value of θ^* such that $\mathrm{Rot}_{\theta^*} \in \mathrm{Rot}(F)$.

4. Prove that if p is taken to be a point of highest rotational symmetry, then

$$\mathrm{Isom}(F)^+ = \{T_1 \circ T_2 : T_1 \in \mathrm{Trans}(F), T_2 \in \mathrm{Point}(F,p)^+\} \stackrel{\text{def}}{=} \mathrm{Trans}(F) \circ \mathrm{Point}(F,p)^+.$$

   (For notational simplicity, you may take assume that $p = 0$.)

5. Deduce that if p is a point of highest rotational symmetry, then one of the following cases must hold:

   (a) $\mathrm{Isom}(F)$ does not contain a reflection or glide-reflection, and $\mathrm{Isom}(F) = \mathrm{Trans}(F) \circ \mathrm{Point}(F,p)$.

   (b) $\mathrm{Point}(F,p)$ contains a reflection, and $\mathrm{Isom}(F) = \mathrm{Trans}(F) \circ \mathrm{Point}(F,p)$.

   (c) $\mathrm{Isom}(F)$ contains a reflection or glide-reflection R, $\mathrm{Point}(F,p)$ does not contain a reflection, and $\mathrm{Isom}(F) = (\mathrm{Trans}(F) \circ \mathrm{Point}(F,p)) \cup (\mathrm{Trans}(F) \circ \mathrm{Point}(F,p) \circ R)$.

   In particular, we can obtain generators for $\mathrm{Isom}(F)$ by taking generators for $\mathrm{Point}(F,p)$ (at most 2 needed), generators for $\mathrm{Trans}(F)$ (exactly 2 needed), and possibly an additional reflection R.

Problem 3. (Characteristic and Order of Finite Fields [AM106])

1. Show that if R is an integral domain of nonzero characteristic p, then every nonzero element of R has additive order p.

2. Use the classification of finite abelian groups to show that if F is a finite field of characteristic p, then the order (i.e. size) of F is p^n for some $n \in \mathbb{N}$.

Problem 4. (Adjoining Square Roots) Which of the following rings are integral domains? fields? Justify your answers.

1. [AM106-A] $\mathbb{Z}_{15}[\sqrt{2}]$. (Elements are of the form $(a + b\sqrt{2})$ with $a, b \in \mathbb{Z}_{15}$, addition defined by $(a + b\sqrt{2}) + (c + d\sqrt{2}) = ((a + c) \bmod 15) + ((b + d) \bmod 15)\sqrt{2}$, and multiplication defined by $(a + b\sqrt{2})(c + d\sqrt{2}) = ((ac + 2bd) \bmod 15) + ((ad + bc) \bmod 15)\sqrt{2}$.)

2. [AM106-A] $\mathbb{Z}_{11}[\sqrt{2}]$. (Defined similarly to previous item.)

3. [AM106-A] $\mathbb{Z}_7[\sqrt{2}]$. (Defined similarly to previous item.)

4. [AM206-A] Characterize when $\mathbb{Z}_n[\sqrt{k}]$ is a field for arbitrary positive integers n and k. Your characterization should take the form of "$\mathbb{Z}_n[\sqrt{k}]$ is a field if and only if n has Property X and the equation '$\cdots = \cdots$' (in one variable x) has no solution in $\mathbb{Z}_n$."