Problem 1. (The Si(111) Reconstructed Face) Attached is a piece of the reconstructed Si(111) face, which is repeated infinitely to form a 2-D crystal \( F \). (This face is obtained by cutting a 3-D silicon crystal along a different plane than the one giving the Si(100) face seen in lecture.)

1. On the attached diagram, draw two vectors that generate the translation lattice of \( F \).
2. Find and mark a point \( p \) of maximal rotational symmetry, and determine the group Point\( (F, p) \).
3. Use the flowchart in Gallian Figure 28.18 to classify Isom\( (F, p) \) among the 17 2-D crystallo-
graphic groups.
4. Using generators for Point\( (F, p) \), determine whether the diffusivity of the Si(111) face is isotropic.

Problem 2. (From Translations and Point Groups to the Full Symmetry Group) Let \( E_2 \) be the 2-dimensional Euclidean group, and \( F : X \to \mathbb{R}^2 \) be a 2-dimensional crystal.

1. Let \( E_2^+ \) denote the set of rotations in \( E_2 \), i.e. the set of isometries of the form \( T(x) = \text{Rot}_\theta x + b \), for \( \theta \in [0, 2\pi) \) and \( b \in \mathbb{R}^2 \). Show that \( E_2^+ \) is a subgroup of \( E_2 \), and that it is of index 2.
2. Let \( \text{Isom}(F)^+ = \text{Isom}(F) \cap E_2^+ \). Show that either \( \text{Isom}(F)^+ = \text{Isom}(F) \) or \( \text{Isom}(F)^+ \) is a subgroup of \( \text{Isom}(F) \) and that it is of index 2. Similarly, for a point \( p \in \mathbb{R}^2 \), if we define Point\( (F, p)^+ = \text{Point}(F, p) \cap E_2^+ \) then Point\( (F, p)^+ \) either equals Point\( (F, p) \) or is a subgroup.
of Point\((F,p)\) of index 2. (Hint: these statements are have nothing to do with geometry, and
generalize to studying the intersection \(H^+\) of arbitrary subgroups \(G^+, H\) of a group \(G\) such
that \([G : G^+] = 2\).)

3. Let \(\text{Rot}(F) = \{\text{Rot}_\theta : \exists b \text{ s.t. } T(x) = \text{Rot}_\theta x + b \text{ is in Isom}(F)\}\). Show that \(\text{Rot}(F)\) is a cyclic
  group generated by \(\text{Rot}_\theta^*\) for the smallest positive value of \(\theta^*\) such that \(\text{Rot}_\theta^* \in \text{Rot}(F)\).

4. Prove that if \(p\) is taken to be a point of highest rotational symmetry, then

\[
\text{Isom}(F)^+ = \{T_1 \circ T_2 : T_1 \in \text{Trans}(F), T_2 \in \text{Point}(F,p)^+\} \overset{\text{def}}{=} \text{Trans}(F) \circ \text{Point}(F,p)^+.
\]

(For notational simplicity, you may take assume that \(p = 0\).)

5. Deduce that if \(p\) is a point of highest rotational symmetry, then one of the following cases
  must hold:
  
  (a) \(\text{Isom}(F)\) does not contain a reflection or glide-reflection, and \(\text{Isom}(F) = \text{Trans}(F) \circ \text{Point}(F,p)\).
  
  (b) \(\text{Point}(F,p)\) contains a reflection, and \(\text{Isom}(F) = \text{Trans}(F) \circ \text{Point}(F,p)\).
  
  (c) \(\text{Isom}(F)\) contains a reflection or glide-reflection \(R\), \(\text{Point}(F,p)\) does not contain a reflection,
      and \(\text{Isom}(F) = (\text{Trans}(F) \circ \text{Point}(F,p)) \cup (\text{Trans}(F) \circ \text{Point}(F,p) \circ R)\).

In particular, we can obtain generators for \(\text{Isom}(F)\) by taking generators for \(\text{Point}(F,p)\)
(at most 2 needed), generators for \(\text{Trans}(F)\) (exactly 2 needed), and possibly an additional
reflection \(R\).

Problem 3. (Characteristic and Order of Finite Fields \([AM106]\))

1. Show that if \(R\) is an integral domain of nonzero characteristic \(p\), then every nonzero element
   of \(R\) has additive order \(p\).

2. Use the classification of finite abelian groups to show that if \(F\) is a finite field of characteristic
   \(p\), then the order (i.e. size) of \(F\) is \(p^n\) for some \(n \in \mathbb{N}\).

Problem 4. (Adjoining Square Roots) Which of the following rings are integral domains?
fields? Justify your answers.

1. \([AM106-A]\) \(\mathbb{Z}_{15}[\sqrt{2}]\). (Elements are of the form \((a + b\sqrt{2})\) with \(a, b \in \mathbb{Z}_{15}\), addition defined by
   \((a + b\sqrt{2}) + (c + d\sqrt{2}) = ((a + c) \text{ mod } 15) + ((b + d) \text{ mod } 15)\sqrt{2}\), and multiplication defined
   by \((a + b\sqrt{2})(c + d\sqrt{2}) = ((ac + 2bd) \text{ mod } 15) + ((ad + bc) \text{ mod } 15)\sqrt{2}\).)

2. \([AM106-A]\) \(\mathbb{Z}_{11}[\sqrt{2}]\). (Defined similarly to previous item.)

3. \([AM106-A]\) \(\mathbb{Z}_{7}[\sqrt{2}]\). (Defined similarly to previous item.)

4. \([AM206-A]\) Characterize when \(\mathbb{Z}_n[\sqrt{k}]\) is a field for arbitrary positive integers \(n\) and \(k\). Your
   characterization should take the form of “\(\mathbb{Z}_n[\sqrt{k}]\) is a field if and only if \(n\) has Property X
   and the equation ‘\(\cdots = \cdots\)’ (in one variable \(x\)) has no solution in \(\mathbb{Z}_n\).”