AM 106: Applied Algebra	**Salil Vadhan**

Lecture Notes 14

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- Reading: Gallian Chs. 10, 27.

1 Homomorphisms

- **Def:** For groups G, H, and mapping $\varphi : G \to H$ is a *homomorphism* if for all $a, b \in G$, we have $\varphi(ab) = \varphi(a)\varphi(b)$.
  - Note: we don't require that φ is one-to-one or onto!
- **Def:** For a homomorphism $\varphi : G \to H$,
  - the *image* of φ is $\mathrm{Im}(\varphi) = \varphi(G) = \{\varphi(g) : g \in G\} \leq H$.
  - the *kernel* of φ is $\mathrm{Ker}(\varphi) = \{g \in G : \varphi(g) = \varepsilon\} \lhd G$.
- **Thm 10.3:** If $\varphi : G \to H$ is a homomorphism, then $G/\mathrm{Ker}(\varphi) \cong \mathrm{Im}(\varphi)$.
  Picture:

- **Examples:**

Domain	Range	Mapping	Homo.?	Image	Kernel
$\mathbb{Z}$	$\mathbb{Z}_n$	$x \mapsto x \bmod n$			
$\mathbb{Z}_n$	$\mathbb{Z}_d$	$x \mapsto x \bmod d$			
$\mathbb{R}^n$	$\mathbb{R}^n$	$x \mapsto Mx$, M a matrix			
$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z}$	$(x, y) \mapsto xy$			
S_n	$\{\pm 1\}$	$\sigma \mapsto \mathrm{sign}(\sigma)$			
$\mathbb{R}$	$\mathbb{C}^*$	$x \mapsto e^{2\pi i x}$			
$\mathbb{Z}_3 \times \mathbb{Z}_5$	$\mathbb{Z}_3$	$(x, y) \mapsto x$			
G	G/N, where $N \lhd G$	$g \mapsto gN$			

- **Properties of Homomorphisms:**
  1. $\varphi(\varepsilon_G) = \varepsilon_H$.
  2. $\varphi(a^{-1}) = \varphi(a)^{-1}$.
  3. order($\varphi(a)$) divides order(a).
- **Properties of Images:**
  1. $\varphi(G)$ is a subgroup of H.

2. G cyclic $\Rightarrow \varphi(G)$ cyclic.
3. G abelian $\Rightarrow \varphi(G)$ abelian.

- **Properties of Kernels:**
  1. $\text{Ker}(\varphi)$ is *normal* subgroup of G.
     - Can prove that K is normal by finding a homomorphism φ s.t. $\text{Ker}(\varphi) = K$.
  2. $\varphi(a) = \varphi(b) \Leftrightarrow b^{-1}a \in \text{Ker}(\varphi) \Leftrightarrow a\text{Ker}(\varphi) = b\text{Ker}(\varphi)$.
  3. φ injective (one-to-one) if and only if $\text{Ker}(\varphi) = \{\varepsilon\}$.

- **Proof of Thm 10.3 ($G/\text{Ker}(\varphi) \cong \text{Im}(\varphi)$):**

2 Isometries

- Motivation: most of the applications of group theory to the physical sciences are through the study of the symmetry groups of physical objects (e.g. molecules or crystals). Understanding the symmetries helps in understanding the objects' physical properties and in determining the structure of the objects from measurements or images.

- Recall that symmetry groups of geometric objects are defined in terms of isometries, so we begin by understanding those.

- **Def:** An *isometry* of $\mathbb{R}^n$ is a function $T : \mathbb{R}^n \to \mathbb{R}^n$ such that for every $x, y \in \mathbb{R}^n$, we have $\|T(x) - T(y)\| = \|x - y\|$.
  - Isometries are always permutations (bijections).
  - The set of isometries of $\mathbb{R}^n$ forms a group under composition, known as the *Euclidean Group E_n*.
  - Isometries preserve angles: $\langle T(x) - T(z), T(y) - T(z)\rangle = \langle x - z, y - z\rangle$.
  - Although most physical objects live in $\mathbb{R}^3$, we'll focus on objects in $\mathbb{R}^2$. Symmetry of 2-D objects is useful in *surface physics*. Everything we'll discuss has generalizations to $\mathbb{R}^3$.

- **Linear-algebraic Description of Isometries:**
  - **Fact:** The isometries of $\mathbb{R}^n$ are exactly the maps of the form $T(x) = Ax + b$, where A is an $n \times n$ orthogonal matrix (i.e., $AA^t = I$, where A^t is the transpose of A) and $b \in \mathbb{R}^n$.
  - In $\mathbb{R}^2$, the possible orthogonal matrices A are:

$$\text{Rot}_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \text{ and } \text{Ref}_\theta = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

    for $\theta \in [0, 2\pi)$. Rot_θ is a clockwise rotation around the origin by angle θ. Ref_θ is a reflection through the axis that is the y-axis rotated clockwise by angle $\theta/2$.

- **Classification of Isometries $T(x) = Ax + b$ of $\mathbb{R}^2$:**

- $A = \mathrm{Rot}_0 = I$: T is a *translation.*
- $A = \mathrm{Rot}_\theta$ for $\theta \in (0, 2\pi)$: T is a clockwise *rotation* by θ degrees about the point $(I - A)^{-1}b$. ($I - A$ is invertible because A has no fixed points.)
- $A = \mathrm{Ref}_\theta$, b orthogonal to the axis ℓ of reflection: T is a *reflection* through the axis $\ell + b/2$.
- $A = \mathrm{Ref}_\theta$, b parallel to axis of reflection: T is a *glide-reflection*:
- $A = \mathrm{Ref}_\theta$, b neither parallel nor perpendicular to axis of reflection: T is a *glide-reflection* along the axis $\ell + b'/2$, where b' is the component of b perpendicular to the axis of reflection.

- **Q:** What are the orders of each of the above elements (in the group of isometries of $\mathbb{R}^2$ under composition)?