AM 106: Applied Algebra **Salil Vadhan**

Lecture Notes 18

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1 Ideals

- Reading: Gallian Ch. 14
- **Goal:** ring-theoretic analogue of normal subgroup, a set of elements we can "mod out" (set to zero) to get a factor ring.
  - Normal subgroups: since $a\varepsilon a^{-1} = \varepsilon$ in every group, we need $aNa^{-1} \subseteq N$ for N to work as an identity element in a factor group G/N.
  - Ideals: since $a \cdot 0 = 0$ in every ring, we need $aI \subseteq I$ for I to work as an identity element in a factor ring R/I.
- **Def:** Let R be a commutative ring with unity. A set $I \subseteq R$ is an *ideal* iff (a) I is a subgroup of R under addition, and (b) for every $a \in I$ and $r \in R$, we have $ar \in I$.
  - Contrast with a *subring* I, where we would only require condition (b) to hold when $r \in I$.
- **Thm 14.2 (Factor Rings):** If R is a commutative ring with unity and $I \subseteq R$ is an ideal, then the additive cosets of I form a ring, denoted R/I, under the operations $(a+I)+(b+I) = (a+b)+I$ and $(a+I)(b+I) = ab+I$
- **Examples and Non-examples:**
  - $\{0\}$.
  - R.
  - Ideals in $\mathbb{Z}$.
  - $R = \mathbb{R}[x]$, $I = \{p(x) : p(11) = 0\}$.

- $R = \mathbb{R}[x]$, $I = \{p(x) : p(11) = 5\}$.

- $R = \mathbb{C}[x]$, $I = \mathbb{Q}[x]$.

- Ideals in a field.

- *Principal ideal* generated by $a \in R$: $\langle a \rangle = \{ra : r \in R\}$. (Which of above ideals are principal?)

- Ideal generated by $a_1, \ldots, a_k$: $\langle a_1, \ldots, a_k \rangle = \{r_1 a_1 + \cdots + r_k a_k : r_1, \ldots, r_k \in R\}$.

- $R = \mathbb{Z}$, $I = \langle m, n \rangle$.

- $R = \mathbb{Q}[x]$, $I = \langle x^2 - 7, x \rangle$.

- $R = \mathbb{Z}[x]$, $I = \langle 17, x \rangle$.

• **Theorem 14.4**: Let R be a commutative ring with unity and I an ideal in R. Then R/I is a field if and only if I is a *maximal ideal.* That is, $I \neq R$ but I is not contained in any ideal of R other than I and R.
Proof:

• **Examples:**

- Maximal Ideals in $\mathbb{Z}$:

- $\langle 17, x\rangle$ vs. $\langle 17\rangle$ and $\langle x\rangle$ in $\mathbb{Z}[x]$.

- There is also a characterization of when R/I is an integral domain (namely, when I is a "prime ideal") but we won't cover it.