AM 106: Applied Algebra **Salil Vadhan**

Lecture Notes 6

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1 More Examples of Groups

- Reading: Gallian Ch. 1,2
- $n \times n$ matrices, with real entries:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}.$$

  - Defines a linear transformation from $\mathbb{R}^n \to \mathbb{R}^n$ by $Av = w$, where $w_i = \sum_j a_{ij} v_j = \langle r_i, v \rangle$ and r_i is i'th row of A.
  - $A + B$ has (i, j)'th entry $a_{ij} + b_{ij}$.
  - AB has (i, j)'th entry $\sum_k a_{ik} b_{kj} = \langle r_i, c_j \rangle$ if r_i is i'th row of A and c_j is j'th column of B.

- Symmetric Group $Sym(S)$
  - Terminology
    - Injection = one-to-one function
    - Surjection = onto function
    - Bijection = one-to-one and onto function
    - Permutation = bijection from a set to itself
  - $Sym(S)$ is the set of all permutations $\pi : S \to S$ under composition. $\pi \circ \tau$ is the permutation defined by $(\pi \circ \tau)(x) = \pi(\tau(x))$.
  - $S_n = Sym(\{1, \ldots, n\})$.
  - **Example:** S_3
  - **Q:** $|S_n| =?$

- Dihedral Group D_n

Notation	Set	Operation	Closure?	Associative?	Identity?	Inverses?	Group?	Commutative?
$M_n(\mathbb{R})$	$n \times n$ real matrices	+ entrywise						
	$n \times n$ real matrices	matrix mult.						
$GL_n(\mathbb{R})$	$n \times n$ invertible real matrices	matrix mult.						
S_n	permutations $[n] \mapsto [n]$	composition						
$Sym(S)$	permutations $S \mapsto S$	composition						
D_n	symmetries of regular n-gon	composition						

- "Symmetries" of regular n-gon, $n \geq 3$.
- D_n is the set of distance-preserving transformations T of the plane such that $T(n\text{-gon}) = n\text{-gon}$.
- Elements of D_n
  - If we label vertices $0, 1, \ldots, n-1$ (representing points in $\mathbb{R}^2$) clockwise, then each element $T \in D_n$ is determined by $T(0)$ and $T(1)$.
  - $\text{Rot}_k(i) = k + i \bmod n$: Clockwise rotation by $(k/n)360°$.
  - $\text{Ref}_k(i) = k - i \bmod n$: Reflection through line at $(k/n)180°$ clockwise from line through vertex 0.
- Generalizes to define symmetries of other geometric objects, eg of tilings, of molecules, and of crystals (cf. Gallian Chs 27–28).

2 Basic Properties of Groups

- Reading: Gallian Ch. 2
- **Thm 2.1 (Identity is Unique):** In every group G, there is only one identity element.
  Proof:

- **Thm 2.3 (Inverses are Unique):** For every group G and every element $a \in G$, there is only one inverse of a in G (typically denoted a^{-1}).
  Proof: similar to uniqueness of the identity.
- Multiplicative Notation for Groups
  - Group operation: $a \cdot b$ or just ab
  - Identity: 1 or e
  - Inverse of a: a^{-1}
  - a multiplied n times: a^n
- Additive Notation for Groups
  - Group operation: $a + b$
  - Identity: 0
  - Inverse of a: $-a$
  - a added n times: na
  - Only used for abelian groups!
- **Thm 2.2 (Left-cancellation and Right-cancellation):** In a group:
  1. $ab = ac \Rightarrow b = c$.
  2. $ba = ca \Rightarrow b = c$.
- **Thm 2.4 (Shoes-Socks Property):** In a group, $(ab)^{-1} = b^{-1}a^{-1}$.
  Proof: omitted

3 Order

- Reading: Gallian, Chapter 3.
- **Def:** The *order* of a group G, denoted $|G|$, is the number of elements in G (possibly ∞).
- **Def:** For a group G and $g \in G$, the *order* of g, denoted $|g|$, is the smallest positive integer n such that $g^n = e$ (or ∞ if no such n exists).
  Example: Orders in S_3

  Example: Orders in $\mathbb{Q}^*$