AM 106: Applied Algebra	**Salil Vadhan**
Lecture Notes 13	
	October 18, 2018

- Reading: Gallian Chapter 9

1 Normal Subgroups

- Motivation:
  - Recall that the cosets of $n\mathbb{Z}$ in $\mathbb{Z}$ $(a+n\mathbb{Z})$ are the same as the congruence classes modulo n $([a]_n)$
  - These form a group under addition, isomorphic to $\mathbb{Z}_n$: $[a+b]_n = [a+b \bmod n]_n$ depends only on $[a]_n$ and $[b]_n$ (and not on the particular choice of coset representatives a and b), so we can define $[a]_n + [b]_n = [a+b]_n$.
  - **Q:** under what conditions on $H \leq G$ do the left-cosets of H form a group under the operation of G?
- **Def:** A subgroup H of G is *normal* iff for every $a \in G$, $aH = Ha$. If this holds, we write $H \lhd G$.
- **Proposition:** For $H \leq G$, the following are equivalent:
  - $H \lhd G$
  - for every $a \in G$, $aHa^{-1} = H$
  - for every $a \in G$, $h \in H$, $aha^{-1} \in H$. That is, if $h \in H$, then all *conjugates* of h are also in H.
- **Examples:**
  - Which subgroups of an abelian group are normal?
  - Which subgroups of S_4 are normal?

  - $A_n \lhd S_n$

2 Factor Groups

- **Thm 9.2:** If $H \lhd G$, then the operation $(aH)(bH) = abH$ on the left-cosets of H is well-defined (does not depend on the choice of coset representatives a, b) and forms a group, denoted G/H (called a *factor group*, the *quotient* of G by H, or "G mod H").

- **Proof:**

- **Examples:**

  - $\mathbb{Z}/n\mathbb{Z}$

  - S_n/A_n

  - $(\mathbb{Z}_3 \times \mathbb{Z}_5)/(\mathbb{Z}_3 \times \{0\})$

  - S_4/H where H is the normal subgroup of size 4.

  - $\mathbb{Z}_n/\langle a \rangle$

  - $\mathbb{R}/\mathbb{Z}$