1 General Properties of Rings, Integral Domains, and Fields

- **General Properties of Rings (Thm 12.1):** In a ring $R$,
  1. For every $r \in R$, $0 \cdot r = 0$.
  2. For every $a, b \in R$, $(-a) \cdot b = -(a \cdot b) = a \cdot (-b)$.

**Proof:**

- **Def:** A *zero-divisor* in a ring $R$ is a nonzero element $a \in R$ such that $ab = 0$ for some nonzero element $b \in R$.

- **Def:** An *integral domain* is a commutative ring with unity that has no zero-divisors.

- **Prop:** Let $R$ be a commutative ring with unity. Then the following are equivalent:
  1. $R$ is an integral domain, and
  2. $R$ satisfies cancellation: if $a, b, c \in R$ satisfy $ab = ac$ and $a \neq 0$, then $b = c$.

**Proof (1⇒2):**

- **Def:** A *unit* in a ring $R$ with unity is an element with a multiplicative inverse.
  - Not to be confused with *unity*, which is the multiplicative identity, 1.
  - $R^*$ is the set of units in $R$, which can be shown to be a group under multiplication, known as the *group of units in $R$*.
  - **Example:** $\mathbb{Z}^* = $  

- **Def:** A *field* $F$ is a commutative ring $R$ with unity such that $F^* = F - \{0\}$. 

• **Prop:** Every field is an integral domain.
  **Proof:**

• **Thm:** Every finite integral domain is a field.
  **Proof:**

• **Def:** For a commutative ring $R$ with unity, the *characteristic* of $R$ is defined as follows. If 1 has finite additive order $n$, then the characteristic of $R$ is defined to be $n$. If 1 has infinite order, then the characteristic of $R$ is defined to be zero.

• **Thm 13.4:** The characteristic of any integral domain is either 0 or prime.
  **Proof:**