AM 106: Applied Algebra **Salil Vadhan**

Lecture Notes 16

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- Reading: Gallian Chs. 12 & 13

1 Rings

- Many common algebraic structures have not just one, but two operations ("addition" and "multiplication") that are related to each other. In the rest of the course, we will study common properties that such pairs of operations have and general phenomena that follow from these properties, and use this theory to understand specific algebraic structures that are useful in applications.

- **Def:** R with two binary operations $+$, $\cdot$ is a *ring* if it has the following properties:
  1. $(R, +)$ is an abelian group.
  2. $\cdot$ is associative.
  3. Distributive Law: $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in R$.

- **Additional Properties:**
  - *Commutative Rings:* $ab = ba$ for all $a, b \in R$
  - *Rings with Unity:* $\exists 1 \in R \setminus \{0\}$ such that $1a = a1 = a$ for all $a \in R$.
  - *Integral Domain:* Commutative ring with unity in which there are no *zero divisors*: if a and b are nonzero, then ab is nonzero. (Equivalently, the ring allows cancellation: $ab = ac \Rightarrow b = c$.)
  - *Fields:* Commutative ring with unity in which every nonzero element has a multiplicative inverse.
  - **Fact (proof next time):** every field is an integral domain

- **Venn Diagram of Properties:**

- **Notation:**
  - 0 = additive identity, 1 = multiplicative identity (if it exists)
  - $-a$ = additive inverse, a^{-1} = multiplicative inverse (if it exists)
  - $na = a + a + \cdots + a$ for $n \in \mathbb{N}$

Set/notation	Addition	Multiplication	Ring?	Commut.?	Unity?	Int. Domain?	Field?
$\mathbb{Z}$	standard	standard					
$\mathbb{C}, \mathbb{Q}, \mathbb{R}$	standard	standard					
$\mathbb{Z}_n$	$+$ mod n	$\cdot$ mod n					
$\mathbb{Q}[x] = \{$polys with rational coeffs$\}$	standard	standard					
$\mathbb{Q}^{\leq d}[x] =$ polys of degree $\leq d\}$	standard	standard					
$M_n(\mathbb{R})$	componentwise	matrix mult					
subsets of $\{1, \ldots, 100\}$	$\cup$	$\cap$					
$L_2(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid \int \lvert f \rvert^2 < \infty\}$	standard	$(f * g)(x) = \int_t f(t)g(x-t)dt$					
$R_1 \times R_2$, R_1, R_2 rings	componentwise	componentwise					
$R[x] = \{$polys with coeffs from $R\}$	standard	standard					
$\{0\}$							
$\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$	standard	standard					
$\mathbb{Q}[i] = \{a + bi : a, b \in \mathbb{Q}\}$	standard	standard					
$\mathbb{Z}_3[i] = \{a + bi : a, b \in \mathbb{Z}_3\}$	standard	standard					