Problem 1. (Homomorphisms from a Cyclic Group, 10pts) How many homomorphisms are there from $\mathbb{Z}_4$ to $S_4$? (Hint: how does $\varphi(x)$ relate to $\varphi(1)$? And what can we say about the order of $\varphi(1)$?)

Problem 2. (Factor Groups and Homomorphisms, 15pts) For each of the following groups $G$ and subsets $H \subseteq G$, determine whether $H$ is a normal subgroup of $G$. If yes, then find a familiar group $G'$ such that $G/H \cong G'$, and prove that $G/H \cong G'$ by giving an appropriate homomorphism from $G$ to $G'$.

1. $G = \mathbb{Z}$, $H = \{\text{odd integers}\}$.
2. $G = \mathbb{Q} \times \mathbb{Q}$, $H = \{(q,q) : q \in \mathbb{Q}\}$.
3. $G = S_5 \times S_5$, $H = \{(\sigma,\sigma) : \sigma \in S_5\}$.
4. $G = GL_n(\mathbb{R})$, $H = \{M \in GL_n(\mathbb{R}) : \det(M) \in \{-1,1\}\}$.
5. $G = \mathbb{Z}_{19}^*$, $H = \{\text{the squares in } G\}$.
6. $G = \mathbb{Z}_{57}^*$, $H = \{\text{the squares in } G\}$.

Problem 3. (The Si(111) Reconstructed Face, 13pts) Attached is a piece of the reconstructed Si(111) face, which is repeated infinitely to form a 2-D crystal $F$. (This face is obtained by cutting a 3-D silicon crystal along a different plane than the one giving the Si(100) face seen in lecture.)

1. On the attached diagram, draw two vectors that generate the translation lattice of $F$.
2. Find and mark a point $p$ of maximal rotational symmetry, and determine the group $\text{Point}(F,p)$.
3. Use the flowchart in Gallian Figure 28.18 to classify $\text{Isom}(F,p)$ among the 17 2-D crystallographic groups.
4. Using generators for $\text{Point}(F,p)$, determine whether the diffusivity of the Si(111) face is isotropic.
1. Let $E_2^+$ denote the set of rotations in $E_2$, i.e. the set of isometries of the form $T(x) = \text{Rot}_\theta x + b$, for $\theta \in [0, 2\pi)$ and $b \in \mathbb{R}^2$. Show that $E_2^+$ is a subgroup of $E_2$, and that it is of index 2.

2. Show that for a group $G$ and two subgroups $H \leq G$ and $G^+ \leq G$ such that $[G : G^+] = 2$, the subgroup $H^+ := H \cap G^+$ satisfies that either $H^+ = H$ or $[H : H^+] = 2$.

Below it will be useful to apply this with $G = E_2$, $G^+ = E_2^+$, and either $H = \text{Isom}(F)$ or $H = \text{Point}(F, p)$. Consequently we define $\text{Isom}(F)^+ = \text{Isom}(F) \cap E_2^+$ and $\text{Point}(F, p)^+ = \text{Point}(F, p) \cap E_2^+$.

3. Let $\text{Rot}(F) = \{\text{Rot}_\theta : \exists b \text{ s.t. } T(x) = \text{Rot}_\theta x + b \text{ is in Isom}(F)\}$. Assuming that $\text{Rot}(F)$ is nontrivial (i.e. contains a rotation other than $\text{Rot}_0$), let $\theta^*$ be the smallest positive number such that $\text{Rot}_{\theta^*} \in \text{Rot}(F)$. Show that $\text{Rot}(F)$ is a cyclic group generated by $\text{Rot}_{\theta^*}$. (As shown in class/section, $\theta^* \in \{\pi/3, \pi/2, 2\pi/3, \pi\}$.)

4. Prove that if $p$ is taken to be a point of highest rotational symmetry (i.e. $\text{Point}(F, p)$ contains $\text{Rot}_{\theta^*} x + b$ for an appropriate choice of $b$), then

$$\text{Isom}(F)^+ = \{T_1 \circ T_2 : T_1 \in \text{Trans}(F), T_2 \in \text{Point}(F, p)^+\} \overset{\text{def}}{=} \text{Trans}(F) \circ \text{Point}(F, p)^+.$$

(For notational simplicity, you may take assume that $p = 0$.)

5. Deduce that if $p$ is a point of highest rotational symmetry, then one of the following cases must hold:

(a) $\text{Isom}(F)$ does not contain a reflection or glide-reflection, and $\text{Isom}(F) = \text{Trans}(F) \circ \text{Point}(F, p)$.

(b) $\text{Point}(F, p)$ contains a reflection, and $\text{Isom}(F) = \text{Trans}(F) \circ \text{Point}(F, p)$.

(c) $\text{Isom}(F)$ contains a reflection or glide-reflection $R$, $\text{Point}(F, p)$ does not contain a reflection, and $\text{Isom}(F) = (\text{Trans}(F) \circ \text{Point}(F, p)) \cup (\text{Trans}(F) \circ \text{Point}(F, p) \circ R)$.

In particular, we can obtain generators for $\text{Isom}(F)$ by taking generators for $\text{Point}(F, p)$ (at most 2 needed), generators for $\text{Trans}(F)$ (exactly 2 needed), and possibly an additional reflection $R$. 

The circles are Silicon atoms, with the size and color indicating depth from the surface. Ignore the dotted lines, numbers, whited-out material, and other imperfections in the diagram.