AM 106: Applied Algebra	**Prof. Salil Vadhan**
Problem Set 6	
Assigned: Fri. Oct. 26, 2018	Due: Fri. Nov. 2, 2018 (11:59pm sharp)

- You must submit your problem sets electronically on the course Canvas site. If you use LaTeX, please submit both the source (`.tex`) and the compiled file (`.pdf`). Name your files `PS6-yourlastname`.
- Aim for clarity and conciseness in your solutions, emphasizing the main ideas over low-level details.

Problem 1. (Homomorphisms from a Cyclic Group, 10pts) How many homomorphisms are there from $\mathbb{Z}_4$ to S_4? (Hint: how does $\varphi(x)$ relate to $\varphi(1)$? And what can we say about the order of $\varphi(1)$?)

Problem 2. (Factor Groups and Homomorphisms, 15pts) For each of the following groups G and subsets $H \subseteq G$, determine whether H is a normal subgroup of G. If yes, then find a familiar group G' such that $G/H \cong G'$, and prove that $G/H \cong G'$ by giving an appropriate homomorphism from G to G'.

1. $G = \mathbb{Z}$, $H = \{\text{odd integers}\}$.
2. $G = \mathbb{Q} \times \mathbb{Q}$, $H = \{(q, q) : q \in \mathbb{Q}\}$.
3. $G = S_5 \times S_5$, $H = \{(\sigma, \sigma) : \sigma \in S_5\}$.
4. $G = GL_n(\mathbb{R})$, , $H = \{M \in GL_n(\mathbb{R}) : \det(M) \in \{-1, 1\}\}$.
5. $G = \mathbb{Z}_{19}^*$, $H = \{\text{the squares in } G\}$.
6. $G = \mathbb{Z}_{57}^*$, $H = \{\text{the squares in } G\}$.

Problem 3. (The Si(111) Reconstructed Face, 13pts) Attached is a piece of the reconstructed Si(111) face, which is repeated infinitely to form a 2-D crystal F. (This face is obtained by cutting a 3-D silicon crystal along a different plane than the one giving the Si(100) face seen in lecture.)

1. On the attached diagram, draw two vectors that generate the translation lattice of F.
2. Find and mark a point p of maximal rotational symmetry, and determine the group $\text{Point}(F, p)$.
3. Use the flowchart in Gallian Figure 28.18 to classify $\text{Isom}(F, p)$ among the 17 2-D crystallographic groups.
4. Using generators for $\text{Point}(F, p)$, determine whether the diffusivity of the Si(111) face is isotropic.

Problem 4. (From Translations and Point Groups to the Full Symmetry Group, 22pts) Let E_2 be the 2-dimensional Euclidean group, i.e., the group of isometries in $\mathbb{R}^2$ under composition. Let $F : \mathbb{R}^2 \to X$ be a 2-dimensional crystal.

1. Let E_2^+ denote the set of rotations in E_2, i.e. the set of isometries of the form $T(x) = \text{Rot}_\theta x + b$, for $\theta \in [0, 2\pi)$ and $b \in \mathbb{R}^2$. Show that E_2^+ is a subgroup of E_2, and that it is of index 2.

2. Show that for a group G and two subgroups $H \leq G$ and $G^+ \leq G$ such that $[G : G^+] = 2$, the subgroup $H^+ := H \cap G^+$ satisfies that either $H^+ = H$ or $[H : H^+] = 2$.

   Below it will be useful to apply this with $G = E_2$, $G^+ = E_2^+$, and either $H = \text{Isom}(F)$ or $H = \text{Point}(F, p)$. Consequently we define $\text{Isom}(F)^+ = \text{Isom}(F) \cap E_2^+$ and $\text{Point}(F, p)^+ = \text{Point}(F, p) \cap E_2^+$.

3. Let $\text{Rot}(F) = \{\text{Rot}_\theta : \exists b \text{ s.t. } T(x) = \text{Rot}_\theta x + b \text{ is in } \text{Isom}(F)\}$. Assuming that $\text{Rot}(F)$ is nontrivial (i.e. contains a rotation other than Rot_0), let θ^* be the smallest positive number such that $\text{Rot}_{\theta^*} \in \text{Rot}(F)$. Show that $\text{Rot}(F)$ is a cyclic group generated by Rot_{θ^*}. (As shown in class/section, $\theta^* \in \{\pi/3, \pi/2, 2\pi/3, \pi\}$.)

4. Prove that if p is taken to be a point of highest rotational symmetry (i.e. $\text{Point}(F, p)$ contains $\text{Rot}_{\theta^*} x + b$ for an appropriate choice of b), then

   $$\text{Isom}(F)^+ = \{T_1 \circ T_2 : T_1 \in \text{Trans}(F), T_2 \in \text{Point}(F, p)^+\} \stackrel{\text{def}}{=} \text{Trans}(F) \circ \text{Point}(F, p)^+.$$

   (For notational simplicity, you may take assume that $p = 0$.)

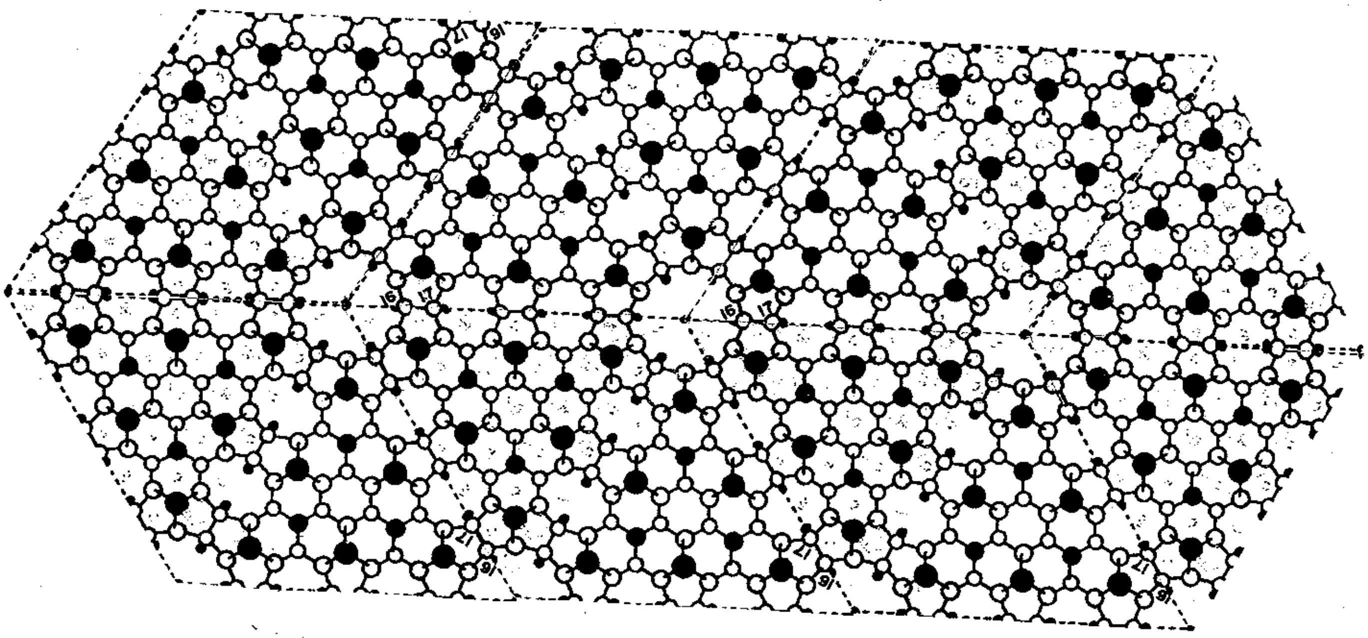
5. Deduce that if p is a point of highest rotational symmetry, then one of the following cases must hold:

   (a) $\text{Isom}(F)$ does not contain a reflection or glide-reflection, and $\text{Isom}(F) = \text{Trans}(F) \circ \text{Point}(F, p)$.

   (b) $\text{Point}(F, p)$ contains a reflection, and $\text{Isom}(F) = \text{Trans}(F) \circ \text{Point}(F, p)$.

   (c) $\text{Isom}(F)$ contains a reflection or glide-reflection R, $\text{Point}(F, p)$ does not contain a reflection, and $\text{Isom}(F) = (\text{Trans}(F) \circ \text{Point}(F, p)) \cup (\text{Trans}(F) \circ \text{Point}(F, p) \circ R)$.

   In particular, we can obtain generators for $\text{Isom}(F)$ by taking generators for $\text{Point}(F, p)$ (at most 2 needed), generators for $\text{Trans}(F)$ (exactly 2 needed), and possibly an additional reflection R.



The Si(111) reconstructed face.

The circles are Silicon atoms, with the size and color indicating depth from the surface.
Ignore the dotted lines, numbers, whited-out material, and other imperfections in the diagram.

Diagram based on article by Brommer et al., Phys. Rev. Lett. A, 68(2), pp. 1356-1359.