

Problem Set 7

Assigned: Fri. Nov. 2, 2018

Due: Fri. Nov. 9, 2018 (11:59pm sharp)

1 Rings

Problem 1. (Rings, 18pts) Which of the following are rings? integral domains? fields? Justify your answers.

1. $\mathbb{Z} \times \mathbb{Z}_2$, with componentwise arithmetic.
2. Polynomials $p \in \mathbb{Q}[x]$ such that $p(0) = 0$, under usual polynomial addition and multiplication.
3. Continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with addition defined as $(f + g)(x) = f(x) + g(x)$ and multiplication defined as $(f \circ g)(x) = f(g(x))$.
4. Functions $f : \mathbb{R} \rightarrow \mathbb{R}$ of the form $f(x) = ax$ where $a \in \mathbb{R}$, with the addition and multiplication defined as in the last example.
5. $\mathbb{Z}_{11}[\sqrt{2}]$. (Elements are of the form $(a + b\sqrt{2})$ with $a, b \in \mathbb{Z}_{11}$, addition defined by $(a + b\sqrt{2}) + (c + d\sqrt{2}) = ((a + c) \bmod 11) + ((b + d) \bmod 11)\sqrt{2}$, and multiplication defined by $(a + b\sqrt{2})(c + d\sqrt{2}) = ((ac + 2bd) \bmod 11) + ((ad + bc) \bmod 11)\sqrt{2}$.)
6. $\mathbb{Z}_7[\sqrt{2}]$. (Defined similarly to previous item.)

Problem 2. (Characteristic and Order of Finite Fields, 14pts)

1. Show that if R is an integral domain of nonzero characteristic p , then every nonzero element of R has additive order p .
2. Use the classification of finite abelian groups to show that if F is a finite field of characteristic p , then the order (i.e. size) of F is p^n for some $n \in \mathbb{N}$.

Problem 3. (Solving Quadratic Equations in Arbitrary Rings, 14pts) Let R be a commutative ring with unity in which 2 is neither a zero-divisor nor equal to zero. Show that the equation $x^2 + bx + c = 0$ for $b, c \in R$ has a solution in R if $b^2 - 4c$ has a square root s in R such that $b + s$ is a multiple of 2 (in R). (Hint: the quadratic formula.) For extra credit, prove the converse.

Problem 4. (EthiCS: Basic vs. Applied Research, 14pts)

1. (For Tue 6/6 lecture) Read excerpts from Effective Altruism website, *Strangers Drowning*, and *A Mathematician's Apology*.
2. Write 1-2 paragraphs in response to the following: What are your career goals, and what do you think justifies that choice of career? Please draw on the reading, the material and discussion for class, or your own research on the effective altruistic website.