AM 106/206: Applied Algebra **Prof. Salil Vadhan**

Quiz II: Practice Problems

- Justify all answers except where otherwise noted. You may use any result proved in lecture, the textbook, or problem sets, as long as you state it clearly.

Problem 1. (Examples of algebraic structures) For each of the following, either give an example or else prove that no such example is possible:

1. A noncommutative ring R.
2. A finite integral domain that is not a field.
3. A group of order 9 with an element $x \neq \varepsilon$ such that $x = x^{-1}$.
4. An infinite ring of characteristic 2.
5. A commutative ring R without unity.

Problem 2. (Complexity of Cyclic Groups) For which of the following problems do we know polynomial-time algorithms, where p is a large prime, g is a generator of $\mathbb{Z}_p^*$, $x \in \mathbb{Z}_{p-1}$, and $y = g^x \bmod p$:

1. Given p, g, and y, compute x.
2. Given p, g, and x, compute y.
3. Given p, g, and y, determine whether or not x is even.

Problem 3. (The Diagonal Subgroup) Let G be a group, and let $\mathrm{diag}(G) = \{(g, g) : g \in G\} \subseteq G \times G$.

1. Show that $\mathrm{diag}(G)$ is a normal subgroup of G if and only if G is abelian. (Hint: conjugate by (h, ε).)
2. In case G is abelian, prove that $(G \times G)/\mathrm{diag}(G) \cong G$.

Problem 4. (Group Isomorphisms) For each one of the following groups, is it isomorphic to the group $G_0 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$? Give a *brief explanation* in each case.

1. $\mathbb{Z}_4 \times \mathbb{Z}_9$
2. $\mathbb{Z}_6 \times \mathbb{Z}_6$
3. $\mathbb{Z}_5^* \times \mathbb{Z}_{10}^*$
4. $\mathbb{Z}_7^* \times \mathbb{Z}_9^*$
5. D_{18}

Problem 5. (Permutation Groups) List all k such that S_5 has an element of order k. Do the same for A_5. (Hint: consider the possible cycle decompositions.)

Problem 6. (Symmetry in Escher) Consider the symmetry group $\mathrm{Isom}(F)$ of the Escher tessellation F at `http://mathstat.slu.edu/escher/index.php/File:Regular-division-75.jpg`. *Ignore shading; treat the white and black lizards as identical.*

1. Draw vectors that generate the translation lattice $\mathrm{Latt}(F)$.
2. Mark a point p of maximal rotational symmetry and determine the point group $\mathrm{Point}(F, p)$ at point p.
3. Among points q in the plane (including $q = p$), what are the possible sizes of the orbit $|\mathrm{orb}_H(q)|$ and stabilizer $|\mathrm{stab}_H(q)|$ under the point group $H = \mathrm{Point}(F, p)$?