Problem 1. (NP-completeness) In the Bounded Halting problem, we are given a pair \((M, t)\) where \(M\) is a Turing machine and \(t\) is an integer and have to decide whether there exists an input (of length at most \(t\)) on which \(M\) halts within \(t\) steps.

Show that Bounded Halting is NP-complete.

Problem 2. (co-NP) Let \(\text{co-NP} = \{L : \overline{L} \in \text{NP}\}\), the class of languages whose complement is in NP.

1. Show that a language \(L\) is complete for \(\text{NP}\) iff \(\overline{L}\) is complete for \(\text{co-NP}\). (Here completeness is with respect to poly-time mapping reductions, aka Karp reductions.)

2. Show that if \(\text{NP} \neq \text{co-NP}\), then \(\text{P} \neq \text{NP}\).

3. Let \(\text{Tautology} = \{\phi : \phi\text{ a boolean formula s.t. } \forall a, \phi(a) = 1\}\). Show that Tautology is co-NP-complete.

Problem 3. (Why Languages?)

1. Show that for every function \(f : \Sigma^* \rightarrow \Sigma^*\), there is a language \(L\) such that \(f\) is computable in polynomial time if and only if \(L \in P\).

2. A search problem is a mapping \(S\) from strings ("instances") to sets of strings ("valid solutions"). An algorithm \(M\) solves a search problem \(S\) if for every input \(x\) such that \(S(x) \neq \emptyset\), \(M\) outputs some solution in \(S(x)\). An NP search problem is a search problem \(S\) such that there exists a polynomial \(p\) and a polynomial-time algorithm \(V\) such that for every \(x, y\):
• $y \in S(x) \Rightarrow |y| \leq p(|x|)$ and
• $y \in S(x) \iff V$ accepts $(x, y)$

It is widely believed that there is no polynomial-time algorithm for integer factorization. Under this assumption and also using the fact that PRIMES is in P, exhibit two NP search problems $S$ and $T$ such that the corresponding languages, $\{x : S(x) \neq \emptyset\}$ and $\{x : T(x) \neq \emptyset\}$, are identical yet $S$ is solvable in polynomial time and $T$ is not.

3. An NP optimization problem is given by a polynomial-time computable objective function $Obj : \Sigma^* \times \Sigma^* \to \mathbb{Q}_{\geq 0}$, where $\mathbb{Q}_{\geq 0}$ is the set of nonnegative rational numbers and $Obj(x, y) = +\infty$ if $|y| > p(|x|)$ for some polynomial $p$. The problem is: given an input $x$, find $y$ minimizing $Obj(x, y)$. An example is the problem of finding the shortest tour in an instance of the Travelling Salesman Problem.

Prove that the following are equivalent:

• $P = NP$
• Every NP search problem can be solved in polynomial time.
• Every NP optimization problem can be solved in polynomial time.