CS 221: Computational Complexity **Prof. Salil Vadhan**

Problem Set 2

Assigned: Wed. Feb. 17, 2010 Due: Thu. Mar. 4, 2010 (5 PM sharp)

- You must *type* your solutions. LaTeX, Microsoft Word, and plain ascii are all acceptable. Submit your solutions *via email* to `cs221-hw@seas.harvard.edu`. If you use LaTeX, please submit both the compiled file (`.ps`) and the source (`.tex`). Please name your files `PS2-yourlastname.*`.
- Strive for clarity and conciseness in your solutions, emphasizing the main ideas over low-level details. Do not despair if you cannot solve all the problems! Difficult problems are included to stimulate your thinking and for your enjoyment, not to overwork you. *'ed problems are extra credit.

Problem 1. (Certificate characterization of NL?) It is tempting to try to characterize **NL** as follows. We say a language L has *logspace-verifiable certificates* if there is a logspace algorithm M and a polynomial p such that $x \in L$ if and only if there exists a string y of length at most $p(|x|)$ such that $M(x, y) = 1$.

Prove that the class of languages with logspace-verifiable certificates is exactly **NP** (and hence is unlikely to equal **NL**). As mentioned in class and shown in the text, if we restrict the verifier to have *one-way access* to the certificate, then we obtain the class **NL**.

Problem 2. (NL-completeness) Prove that 2SAT is **NL**-complete. (Hint: To prove that it is in **NL**, show that the satisfiability of ϕ can be determined from the answers to polynomially many PATH questions involving the directed graph G_ϕ that includes edges $(\neg x, y)$ and $(\neg y, x)$ for every clause $(x \vee y)$ in ϕ.)

Problem 3. (A PSPACE-complete game) In the World Domination game, two players are given an undirected graph of initially unoccupied countries, a score for each country, and an initial score for each player. They take turns to take up an unoccupied country, provided that it is not adjacent to the other player's existing territory (countries), to avoid frictions. Each country is associated with a score, and the player with higher total score wins.

Show that deciding whether the first player has a winning strategy in the World Domination game (given the country graph and scores in binary) is **PSPACE**-complete.

Problem 4. (Collapsing the Hierarchy) Show that $\mathbf{\Sigma}_k^p = \mathbf{\Pi}_k^p$ implies $\mathbf{PH} = \mathbf{\Sigma}_k^p$, i.e. the polynomial hierarchy collapses to the kth level.

Problem 5. (More Time-Space Tradeoffs for SATISFIABILITY) The time-space tradeoffs done in class optimize the space lower bound ($n^{1-\epsilon}$) while giving a relatively weak time lower bound ($n^{1+o(1)}$). On this problem, you'll do the opposite, giving a time lower bound of $n^{1.41}$ while giving a weaker space lower bound ($n^{o(1)}$).

Do not worry about constructibility of the time and space bounds on this problem.

1. Show that for every $T(n) \geq n^2$, $\mathbf{TISP}(T, T^{o(1)}) \subseteq \Sigma_2\mathbf{TIME}(T^{1/2+o(1)})$.

2. Use the above to prove that SAT $\notin \mathbf{TISP}(n^c, n^{o(1)})$ for any $c < \sqrt{2}$. (Hint: Use a NONdeterministic-time Hierarchy Theorem.)