Problem 1. (circuit complexity of a threshold function) Consider the threshold function \( \text{Th}_2(x_1, \ldots, x_n) \), defined to be 1 iff at least two of the input variables are 1.

1. Prove that \( \text{size}_{\{\land, \lor, \neg}\}(\text{Th}_2) \leq 4n + O(1) \). (Recall that our measure of circuit size includes the input variables.)

2. Prove that \( \text{size}_{B_2}(\text{Th}_2) \geq 3n - O(1) \), where \( B_2 \) is the full binary basis. (Hint: show that if two variables are inputs to some binary gate, then at least one of them must be used elsewhere in the circuit.)

Problem 2. (branching programs) A branching program over variables \( \{x_1, \ldots, x_n\} \) is a directed acyclic graph where every node is labelled with a variable \( x_i \), or is labelled with an output in \( \{0, 1\} \). Variable nodes are required to have outdegree 2 and output nodes must have outdegree 0. The two edges leaving every variable node are also labelled 0 and 1. One of the nodes is designated as the start node. Such a branching program defines a function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \), where \( f(\alpha) \) is defined as follows. We begin at the start node, then follow the path determined by taking the outgoing edge from each variable node \( v \) according to the value \( \alpha \) assigns to the variable labelling \( v \). Eventually we reach an output node, and set \( f(\alpha) \) to be the value at that node.

1. Characterize the class of languages decidable by polynomial-sized branching programs in terms of one of the complexity classes we have seen, augmented with advice.

2. A branching program has width \( w \) if its nodes can be partitioned into layers \( L_1, L_2, \ldots \) each of size up to \( w \), such that every edge leaving a node in layer \( L_i \) leads to a node in \( L_{i+1} \).
   
   Show that every language decidable by a constant-width, polynomial-sized branching program is in \( \mathbf{NC}^1 \). (Barrington’s Theorem says that the converse is also true, giving a surprising alternate characterization of \( \mathbf{NC}^1 \).)
Problem 3. (circuit lower bounds for high classes)

1. Prove that $\text{EXPSPACE} \not\subseteq \text{SIZE}(2^n/2n)$.

2. Prove that for every constant $c$, $\text{PH} \not\subseteq \text{SIZE}(n^c)$.

3. Prove that for every constant $c$, $\Sigma^p_2 \not\subseteq \text{SIZE}(n^c)$.

Recall that the best circuit lower bound we have for a function in $\text{NP}$ is only $6n - o(n)$.

Problem 4. (refined hierarchy theorem for circuit size*) In Arora–Barak (Thm 6.22), a hierarchy theorem for circuit size is proven, showing that a polynomial or even multiplicative factor in circuit size allows computing more functions. Tighten this hierarchy theorem as much as you can; the amount of extra credit will depend on how tight a hierarchy theorem you get.

Problem 5. (different models of randomized computation) Suppose we modify our model of randomized computation to allow the algorithm to obtain a random element of $\{1, \ldots, m\}$ for any number $m$ whose binary representation it has already computed (as opposed to just allowing it access to random bits). Show that this would not change the classes $\text{BPP}$, $\text{RP}$, and $\text{ZPP}$.