1 Overview

In our previous lecture we discussed criteria for testing whether an LP is infeasible and whether it is unbounded. We then proved Weierstrass theorem which gives us a sufficient (though not necessary) condition for testing when an optimization problem has an optimal solution.

Today we will give a characterization of optimal solutions of linear optimization problem. We’ll show that every LP has an optimal solution which is an extreme point. We will begin with a brief introduction of the celebrated Knapsack problem from combinatorial optimization, and discuss its connection to the concept of extreme points.

2 The Knapsack Problem

The Knapsack problem is a well-studied problem in combinatorial optimization, defined as follows.

**Problem 1 (Knapsack).** Given $n$ items $a_1, \ldots, a_n$ each $a_i$ with cost $c_i \in \mathbb{R}$ and value $v_i \in \mathbb{R}$, and a budget $B \in \mathbb{R}$, the goal in the knapsack problem is to find a subset of items $S$ s.t. $S \in \text{argmax}_{T \subseteq \{a_1, \ldots, a_n\}} \sum_{i \in T} v_i \text{ where } F := \{T \subseteq \{a_1, \ldots, a_n\} : \sum_{i \in T} c_i \leq T\}$.

The knapsack problem can also be written as a solution to an integer program with variables $x_1, \ldots, x_n$, and vectors $v = (v_1, \ldots, v_n), c = (c_1, \ldots, c_n)$.

\[
\begin{align*}
\text{max} & \quad v^T x \\
\text{s.t.} & \quad c^T x \leq B \\
& \quad x_i \in \{0, 1\} \quad \forall i \in [n]
\end{align*}
\]

The knapsack problem is well-known to be NP-hard. Informally, what this means is that we strongly believe that it is computationally intractable to find an algorithm which obtains the optimal solution for this problem on general instances. In the third part of the course we will discuss computational complexity formally, and define this concept precisely. For our purposes now, we can be satisfied with this informal definition.

Let’s now define another problem: Knapsack-LP which will be defined exactly as the problem above, except that we’ll replace condition (3) with the condition: $x_i \in [0, 1], \forall i \in [n]$. Although we still haven’t learned how to solve LPs in general, the structure of the Knapsack-LP problem is simple enough so that we can characterize what optimal solutions look like.
An optimal solution for Knapsack-LP. Observe that our objective is to find a solution at cost \( B \) which is the densest solution possible, i.e. the solution which has the highest value per cost. From this perspective, let’s relabel the items according to density, i.e. \( v_1/c_1 \geq v_2/c_2 \geq \ldots v_n/c_n \), and let’s define \( k \) to be the largest index \( t \) for which \( \sum_{i=t}^{n} c_i \leq B \), when the items are labeled according to their order of density.

We now claim that the optimal solution for Knapsack-LP is to simply take the densest items that we can fit under the budget, and use the remainder of the budget to add the largest fraction of the next densest item, without going over budget.

**Claim 2.** For \( k \) defined as above, the optimal solution to Knapsack-LP is \( x_1 = x_2 = \ldots = x_k = 1 \) and \( x_{k+1} = \frac{B - \sum_{i=1}^{k} c_i}{c_{k+1}} \).

The proof for the above claim follows the fact that the solution above maximizes density under the budget \( B \): for some \( i \leq k + 1 \), if we reduce \( x_i \) and instead increase the \( x_j \) for some \( j \geq k + 1 \), the solution cannot be have a larger value since \( v_i/c_i \geq v_j/c_j \).

The optimal solution we found has an interesting property: it lies on the edge of the polytope which defines the feasible region \((c^T x \leq B)\). But moreover, this solution is on some extreme point of the polytope: there are no points in \( P = \{ x \in \mathbb{R}^n : c^T x \leq B \} \) which can be used to express a convex combination of this solution.

An approximation algorithm for Knapsack. Consider the following algorithm: Select either \( \{a_1, \ldots, a_k\} \) or \( \{a_{k+1}\} \), whichever one has the largest value. Observe that this is a feasible solution to the Knapsack problem – all \( x_i \in \{0, 1\} \) and the cost of the solution is under budget.

How good is this solution? Let \( OPT_{\text{Knapsack}} \), \( OPT_{\text{Knapsack-LP}} \) be the value of the optimal solution to Knapsack and Knapsack-LP, respectively. Then:

\[
OPT_{\text{Knapsack}} \leq OPT_{\text{Knapsack-LP}} \leq \sum_{i=1}^{k} v_i + \frac{B - \sum_{i=1}^{k} c_i}{c_{k+1}} \cdot v_{k+1} \leq \sum_{i=1}^{k+1} v_i \leq 2 \max \left\{ \sum_{i=1}^{k} v_i, v_{k+1} \right\}
\]

Where \( OPT_{\text{Knapsack}} \leq OPT_{\text{Knapsack-LP}} \) is simply due to the fact that a solution to Knapsack is also a solution to Knapsack-LP. Therefore, the simple algorithm above gives us at least half of the value of the optimal solution to the knapsack problem. This is called an approximation algorithm, and the last third of this course on combinatorial optimization we will discuss these concepts at depth. We have therefore have the following result:

**Corollary 3.** For the knapsack problem, there is a simple polynomial-time algorithm which finds a solution with value that is at least half of that of the optimal solution.

3 Extreme Points

The optimal solution in the Knapsack-LP example was an extreme point.

**Definition 4.** Let \( C \subseteq \mathbb{R}^n \) be a non-empty, closed convex set. Then \( \bar{x} \) is an extreme point of \( C \) if there are no two points \( x^1, x^2 \in C \) and \( \lambda \in (0, 1) \) s.t. \( \bar{x} = \lambda x^1 + (1 - \lambda) x^2 \).
Not every polyhedron has extreme points. For example, half spaces. So when does a set have an extreme point?

**Theorem 5.** Let $C \subseteq \mathbb{R}^n$ be a non-empty, closed, convex, set. Then, $C$ has an extreme point if and only if $C$ does not contain a line.

*Proof.* Let $\bar{x}$ be an extreme in $C$. We will show that $\bar{x}$ does not contain a line. Assume, for purpose of contradiction that $C$ contains a line, i.e. $\exists \bar{x} \in C$ s.t. $\{x + \alpha d : \alpha \in \mathbb{R}, d \in \mathbb{R}^n\} \subseteq C$.

For any positive integer $n$, define $x^n = (1 - \frac{1}{n})\bar{x} - \frac{1}{n}(x + nd)$. Since $C$ is convex, $x^n \in C$, for all $n \in \mathbb{N}$. Since $C$ is closed then:

$$\lim_{n \to \infty} x^n = \lim_{n \to \infty} \bar{x} + d - \frac{1}{n}(x - \bar{x}) = \bar{x} + d \in C.$$  

Similarly, $\bar{x} - d \in C$. But then, $\bar{x}$ is a convex combination of two points in $C$ since $\bar{x} = \frac{1}{2}(x + d) + \frac{1}{2}(x - d) \in C$. Contradiction to $\bar{x}$ being an extreme point.

In the other direction, let’s assume that $C$ does not contain a line, and we’ll show that $C$ has an extreme point. We will show this by induction on the dimension of $C$. If $C \subseteq \mathbb{R}$, the claim trivially holds. Let’s assume this claim holds for $C \subseteq \mathbb{R}^{n-1}$ and we will show this holds for $C \subseteq \mathbb{R}^n$.

In $\mathbb{R}^n$, since $C$ does not contain the line, we know it has a boundary point, $x$. Let $H_x$ be the supporting hyperplane at $x$, that is: $H_x = \{z \in \mathbb{R}^n : a^Tz = \alpha\}$ for some $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, and $a^Ty \leq a^Tx$ for every point $y \in C$. Since $C \cap H_x \subseteq \mathbb{R}^{n-1}$ and by the inductive hypothesis, $C \cap H_x$ has an extreme point, $\bar{x}$.

We will now show that $\bar{x}$ is also an extreme point in $C^1$. Let $x^1, x^2 \in C$ and $\lambda \in (0, 1)$ s.t. $\bar{x} = \lambda x^1 + (1 - \lambda)x^2$. Then:

$$a^T\bar{x} = \lambda a^Tx^1 + (1 - \lambda)a^Tx^2.$$

Since $x^1, x^2 \in C$ and $H_x$ is the supporting hyperplane, it is necessarily the case that $a^Tx^1 \leq a^T\bar{x}$ and $a^Tx^2 \leq a^T\bar{x}$. But since $c^T\bar{x} = \lambda c^Tx^1 + (1 - \lambda)c^Tx^2$, this necessarily implies that $x^1 = x^2 = \bar{x}$, or in other words that $\bar{x}$ is an extreme point in $C$.

**Theorem 6.** Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, and consider the LP: $\max\{c^Tx : x \in P\}$. If $P$ has an extreme point, and the LP has an optimal solution, then the LP has an optimal solution which is an extreme point in $P$.

*Proof.* Let $\alpha^*$ be the value of the optimal solution and let $Q$ be the set of optimal solutions, i.e. $Q = \{x \in P : c^Tx = \alpha^*\}$. Since $P$ has an extreme point, it necessarily means that it does not contain a line. Since $Q \subseteq P$ it doesn’t contain a line either, hence, $Q$ contains an extreme point $\bar{x}$.

Similar to the previous proof, we will now show that $\bar{x}$ is also an extreme point in $P$.

Let $x^1, x^2 \in C$ and $\lambda \in (0, 1)$ s.t. $\bar{x} = \lambda x^1 + (1 - \lambda)x^2$. Then:

\[\text{Note that although that it is obvious that } \bar{x} \in C, \text{ it is not obvious that it is an extreme point in } C, \text{ even though it is an extreme point in } C \cap H_x\]
\[ c^T \bar{x} = \lambda c^T x^1 + (1 - \lambda) c^T x^2. \]

Since \( x^1, x^2 \in P \) and \( \alpha^* \) is the optimal solution in \( P \), it is necessarily the case that \( c^T x^1 \leq c^T \bar{x} = \alpha^* \) and \( c^T x^2 \leq a^T \bar{x} = \alpha^* \). But since \( c^T \bar{x} = \lambda a^T x^1 + (1 - \lambda) a^T x^2 \), this necessarily implies that \( x^1 = x^2 = \bar{x} \), implying that \( \bar{x} \) is an extreme point in \( P \). \( \square \)