1 Overview

In our previous lecture we explored the concept of duality which is the cornerstone of Optimization Theory. The goal of today’s lecture is to explore the consequences of duality in two fields where optimization is ubiquitous: Game Theory and Learning Theory.

2 A primer in Game Theory

We will focus on two-player games which constitute a nice introduction to the key concepts of Game Theory. A two-player game is defined by:

- a set of two players
- set of strategies $M = \{1, \ldots, m\}$ and $N = \{1, \ldots, n\}$ for player 1 and player 2 respectively.
- payoff matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{m \times n}$ for player 1 and player 2 respectively.

The signification of the payoff matrices is the following: when player 1 plays action $i \in \{1, \ldots, m\}$ and player 2 plays action $j \in \{1, \ldots, n\}$, $A_{ij}$ is the gain of player 1 and $B_{ij}$ is the gain of player 2.

Example 1. A famous example of two-player games is the Battle of the sexes. This game models a situation where a couple is trying to agree on a program for the evening. The sets of strategies are the same for the husband and the wife: watch a football game or go to the opera. The couple would prefer to do the same activity, but the husband would prefer to watch the football game while the wife would prefer to go to the opera. Possible payoffs for this situation could be the following:

<table>
<thead>
<tr>
<th>Opera</th>
<th>Football</th>
</tr>
</thead>
<tbody>
<tr>
<td>Opera</td>
<td>(3, 2)</td>
</tr>
<tr>
<td>Football</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

Figure 1: Battle of the sexes game. The rows represent the strategies of the wife and the columns represent the strategies of the husband. In this compact representation, we combine both payoff matrices into one table. The upper left cell of this table should be read like this: when the wife and the husband both agree to go to the opera, the gain of the wife is 3 and the gain of the husband is 2.
When analyzing a game, one would like to know which combination of strategies are likely to be chosen by the players. Assuming that the players are rational, i.e. utility maximizers, the notion of Nash equilibria follows naturally.

**Definition 2.** A strategy profile \((i, j) \in M \times N\) is called a Nash equilibrium if the strategy played by each player is optimal given the strategy chosen by the other player:

- \(A_{ij} \geq A_{kj}\) for all \(k \in M\).
- \(B_{ij} \geq B_{ik}\) for all \(k \in N\).

**Example 3.** In the battle of sexes game introduced above, there are two nash equilibria: (Opera, Opera) and (Football, Football). (Opera, Football), for example, is not a Nash equilibrium since the wife could increase her gain from 1 to 2 by choosing Football instead of Opera.

**Example 4.** Another famous example of two-player games is the prisoners’ dilemma. In this problem, two prisoners are in solitary confinement. The police is convinced that both prisoners are guilty but does not have enough evidence to convict them to a long prison sentence. Thus, the police interrogates the prisoners separately. Each prisoner can either stay silent (not reveal anything to the police) or betray the other. If only one of the prisoners betrays the other, then he will be set free and the other will be convicted to a long prison sentence. However, if they betray each other, both prisoners will serve a long prison sentence. Possible payoffs modeling this situation could be the following:

<table>
<thead>
<tr>
<th></th>
<th>Silent</th>
<th>Betray</th>
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<tbody>
<tr>
<td>Silent</td>
<td>((-1, -1))</td>
<td>((-10, 0))</td>
</tr>
<tr>
<td>Betray</td>
<td>((0, -10))</td>
<td>((-8, -8))</td>
</tr>
</tbody>
</table>

A close inspection of this table shows that there is only one Nash equilibrium: (Betray, Betray). Indeed, in all other cases, a silent prisoner can always increase his gain (or reduce his loss) by choosing to betray instead of staying silent.

In this case, the Nash equilibrium is not optimal for the prisoners: they could be both better off by staying silent. This suboptimality is a consequence of the nature of the game: both prisoners choose their action without knowing what the other prisoner does.

### 3 Mixed strategies, zero-sum games and the minimax theorem

When studying games, a more general notion of strategies can also be defined: mixed strategy. A mixed strategy is an assignment of a probability to each (pure) strategy so that. A mixed strategy allows the player to choose a pure strategy at random.

Formally, a mixed strategy for player 1 over the set of pure strategies \(M = \{1, \ldots, m\}\) is a probability vector \(x \in \Delta_M = \{u \in [0,1]^m : \sum_{i=1}^m u_i = 1\}\). A mixed strategy for player 2 is defined in a similar way. Note that a pure strategy can be seen as a degenerate case of mixed strategy where a single probability is set to 1 and all the others are set to 0.
The notion of gain and Nash equilibrium can be extended to this generalized notion of strategy. If player 1 and player 2 have mixed strategies \( x \in \Delta_M \) and \( y \in \Delta_N \) respectively, their gains are defined as the expected gain when they both play selecting pure strategies at random according to \( x \) and \( y \). Formally, the gain of player 1 will be:

\[
\sum_{(i,j) \in M \times N} x_i y_j A_{ij} = x^T A y
\]

and similarly for player 2 by replacing \( A \) with \( B \).

The notion of Nash equilibria is also extended using this expected gain.

**Definition 5.** A profile of mixed strategies \( x \in \Delta_M \) and \( y \in \Delta_N \) is a mixed Nash equilibrium iff:

- \( x^T A y \geq u^T A y \) for all \( u \in \Delta_M \).
- \( x^T B y \geq x^T B v \) for all \( v \in \Delta_N \).

**Example 6.** The penalty kick game models the situation in soccer where a player is about to kick a penalty: usually the goalie stands at the middle of the goal line and the player choose to aim either to the left or to the right of the goal. Because the player stands very close to the goal, the goalie has to decide on a direction in advance and starts diving in this direction even before the player hits the ball. If the goalie dives in the direction chosen by the player, he saves the shot, otherwise he fails. Possible payoffs modeling this situation could be:

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left</td>
<td>(+1, -1)</td>
<td>(-1, +1)</td>
</tr>
<tr>
<td>Right</td>
<td>(-1, +1)</td>
<td>(+1, -1)</td>
</tr>
</tbody>
</table>

where the rows represent the dive direction chosen by the goalie and the columns represent the kick direction chosen by the player. Note that this game has no pure Nash equilibria: in any situation the player with gain \(-1\) can increase his gain by changing his strategy. Intuitively, this simply expresses that there is always a better strategy in retrospect for the losing player. However, this game has one mixed Nash equilibrium: \((1/2, 1/2), (1/2, 1/2)\). When both players choose a direction uniformly at random, this leads to an expected gain of 0 for both of them. Any other choice of probabilities would lead to an unbalancedness either to the left or to the right which could be exploited by the other player.

The fact that the penalty kick game has a mixed Nash equilibrium is not surprising. It is in fact a consequence of a general theorem. Remark that in the penalty kick game, the gain of the kicker is exactly the opposite of the gain of the goalie. More generally we have:

**Definition 7.** A two-player game is a zero-sum game iff \( B = -A \) with \( A \) and \( B \) the payoffs matrices of the players.

Zero-sum game are sometimes called strictly competitive in that the gain of a player is exactly the loss of the other(s) player(s): cooperation is useless in these games. We are now ready to state the main theorem:
**Theorem 8.** For every two-player zero-sum game, there exists a mixed Nash equilibrium.

This theorem is a simple consequence of the following theorem which is itself a simple consequence of the strong duality theorem.

**Theorem 9** (von Neumann’s minimax theorem). We have:

$$
\max_{x \in \Delta_M} \min_{y \in \Delta_N} x^T Ay = \min_{y \in \Delta_N} \max_{x \in \Delta_M} x^T Ay
$$

**Proof.** First remark that for a given $x \in \Delta_M$:

$$
\min_{y \in \Delta_N} x^T Ay = \min_{j \in N} (x^T A)_j
$$

hence, the left-hand side of the minimax theorem can be written as the following LP:

$$
\max t
\text{ s.t. } t \leq (x^T A)_j, \ j \in N
x \geq 0
\sum_{i \in M} x_i = 1
$$

Similarly, the right-hand side can be written as the following LP:

$$
\min s
\text{ s.t. } s \geq (Ay)_i, \ i \in M
y \geq 0
\sum_{j \in N} y_j = 1
$$

and both LPs are dual from each other. The theorem then simply follows from the strong duality theorem.

Let us now see how to prove the existence of mixed Nash equilibria in zero-sum two-player games.

**Proof.** Let us consider a zero-sum two-player game with $A$ the pay-off matrix for player $A$. Let us write $v = \max_{x \in \Delta_M} \min_{y \in \Delta_N} x^T Ay = \min_{y \in \Delta_N} \max_{x \in \Delta_M} x^T Ay$ and consider $x^* \in \arg \max_{x \in \Delta_M} \min_{y \in \Delta_N} x^T Ay$ and $y^* \in \arg \min_{y \in \Delta_N} \max_{x \in \Delta_M} x^T Ay$. Then we claim that $(x^*, y^*)$ is a mixed Nash equilibrium. Indeed, it follows from the definition of $x^*$ and $y^*$ that:

- $x^T Ay^* \leq v \leq x^{*T} Ay^*$ for all $x \in \Delta_M$.
- $x^{*T} Ay \geq v \geq x^{*T} Ay^*$, i.e. $x^{*T} By \leq x^{*T} By^*$ for all $y \in \Delta_N$.

which concludes the proof.
4 Learning and Boosting

We now study an application of duality to learning. More precisely we consider a classification problem over a space $\mathcal{X}$ for which we are given a set of hypothesis (possible classifiers) $\mathcal{H} = \{h : \mathcal{X} \rightarrow \{0, 1\}\}$. The data is sampled from an unknown distribution $q$ over $\mathcal{X}$.

The weak learning assumption states that the set of hypothesis $\mathcal{H}$ is good in the following sense: for any distribution $q$ there exists a hypothesis $h \in \mathcal{H}$ which is wrong less than half the time (i.e is better than a uniformly random classifier). Formally:

$$\exists \gamma, \forall q, \exists h \in \mathcal{H}, \mathbb{P}_{x \sim q}[h(x) \neq c(x)] \leq \frac{1}{2} - \frac{\gamma}{2}$$

where $c(x)$ is by definition the correct class (the true answer) of $x \in \mathcal{X}$.

Surprisingly the weak learning assumption implies something much stronger: it is possible to combine the classifiers in $\mathcal{H}$ to construct a classifier which is always right. This is known as strong learning. This is a consequence of the minimax theorem.

**Theorem 10.** Let $\mathcal{H}$ be a set of hypothesis satisfying the weak learning assumption, then there exists a distribution $p$ on $\mathcal{H}$ such that the weighed majority classifier:

$$c_p(x) := \begin{cases} 1 & \text{if } \sum_{h \in \mathcal{H}} p_h h(x) \geq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

is always correct, i.e, $c_p(x) = c(x)$ for all $x \in \mathcal{X}$.

**Proof.** All we have to do is to find the probability $p$, that is, weights to assign to each classifier $h \in \mathcal{H}$. Let us define the matrix $M \in \{-1, +1\}^{\mathcal{X} \times |\mathcal{H}|}$ such that $M_{ij}$ is +1 when classifier $h_j$ is wrong on data $x_i$ and $-1$ otherwise. We have $M_{ij} = 2\delta_{h_j(x_i) \neq c(x_i)} - 1$, hence (1) can be written more compactly as: $q^T M e_j \leq -\gamma$, where $e_j$ is the $j$-th basis vector of $\mathbb{R}^{|\mathcal{H}|}$. We know that we can find such a $j$ for all $q$, i.e:

$$\min_q \max_j q^T M e_j = \min_q \max_p q^T M p \leq -\gamma$$

where $q$ is a distribution on $\mathcal{X}$ and $p$ is a distribution on $\mathcal{H}$ and where the equality follows from the fact that the basis vectors are the basic feasible solutions of the standard simplex. By the minimax theorem:

$$\max_p \min_q q^T M p = \min_i e_i^T M p \leq -\gamma$$

i.e, there exists $p$ a distribution over $\mathcal{H}$ such that for all $i$, $(M p)_i \leq -\gamma$. This implies that the weighted classifier defined in the statement of the theorem is always correct. $\square$