Submodular Optimization under Noise

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Abstract

We consider the problem of maximizing monotone submodular functions under noise, which to the best of our knowledge has not been studied in the past. There has been a great deal of work on optimization of submodular functions under various constraints, with many algorithms that provide desirable approximation guarantees. However, in many applications we do not have access to the submodular function we aim to optimize, but rather to some erroneous or noisy version of it. This raises the question of whether provable guarantees are obtainable in presence of error and noise. We provide initial answers, by focusing on the question of maximizing a monotone submodular function under cardinality constraints when given access to a noisy oracle of the function. We show that:

- For a cardinality constraint $k \geq 2$, there is an approximation algorithm whose approximation ratio is arbitrarily close to $1 - 1/e$;
- For $k = 1$ there is an approximation algorithm whose approximation ratio is arbitrarily close to $1/2$ in expectation. No randomized algorithm can obtain an approximation ratio better than $1/2 + o(1)$ in expectation;
- If the noise is adversarial, no non-trivial approximation guarantee can be obtained.
1 Introduction

In this paper we study the effects of error and noise on submodular optimization. A function \( f : 2^N \rightarrow \mathbb{R}_+ \) defined on a ground set \( N \) of size \( n \) is submodular if:

\[
f(S \cup T) \leq f(S) + f(T) - f(S \cap T).
\]

Equivalently, submodularity can be defined as a natural diminishing returns property: for any \( S \subseteq T \subseteq N \) and \( a \in N \setminus T \) the function is submodular if:

\[
f_S(a) \geq f_T(a),
\]

where \( f_A(B) = f(A \cup B) - f(A) \) for any \( A, B \subseteq N \). In general, submodular functions may require a representation that is exponential in the size of the ground set, in which case the natural assumption is that we are given access to a value oracle which given a set \( S \) returns \( f(S) \). It is well known that submodular functions admit desirable approximation guarantees and are heavily used in applications such as market design, data mining, and machine learning (see related work section for further discussion). Before defining error and noise formally, we can consider the following example.

Example: maximizing coverage with error. In the maximum-coverage problem we are given a family of sets that cover a universe of items, and the goal is to select a fixed number of sets whose union is maximal. This classic problem is an example of maximizing a monotone \((S \subseteq T \implies f(S) \leq f(T))\) submodular function under a cardinality constraint. It is well known that the celebrated greedy algorithm which iteratively adds the set which includes the largest number of items that have not yet been covered provides a \( 1 - 1/e \) approximation guarantee \([65]\) and that this guarantee is optimal \([28]\). But imagine that the algorithm cannot estimate the underlying coverage of the sets exactly. Instead, the algorithm queries an oracle that has some error. For a concrete example, consider the instance illustrated in Figure 1. If we apply the greedy algorithm with the oracle described in the example, it will begin by selecting a set that covers an element in \( A \), and rather than continuing to select sets from \( B \), the algorithm will select elements from \( A \) until it terminates. The approximation ratio in this case would be linear in the size of the input. In this example, for every set queried, the oracle is guaranteed to return a value that is at least \( 2/3 \) of the underlying objective value, which may at a first glance seem like an excessive error. Note however that for any \( \epsilon > 0 \) the example can be replicated by planting not 2 but \( 1/\epsilon \) items in \( A \), and still the greedy algorithm could only guarantee an \( O(1/k) \)-approximation when choosing \( k \) elements.

The greedy algorithm is heavily used in the theory and applications of submodular optimization, which can make its lack of robustness to small errors somewhat alarming. Since submodular functions can be exponentially representative, it may be reasonable to assume that there are cases where one faces some error in their evaluation. In market design where submodular functions often model agents’ valuations for goods, it seems reasonable to assume that agents do not precisely know their valuations. Even with compact representation, evaluation of a submodular function may be prone to error. In machine learning and data mining the submodular objective functions are often learned from data, and may be subject to error. One recent line of work in machine learning theory, for example, seeks to learn submodular functions by observing samples and returning some surrogate function which is a constant-factor approximation of the submodular function \([39, 7, 6, 3, 34, 35, 24, 25, 33, 36, 5]\). In general, as we consider optimization over large data sets, it seems plausible that there are domains in which the objective is not be known precisely.

Can we retain desirable approximation guarantees in the presence of error?
Figure 1: An illustration of an instance of max-cover for which the greedy algorithm fails with access to an oracle with error. In the instance above there is one family of sets $A$ depicted on the left where all sets cover the same two items, and another family of disjoint sets $B$ that each cover a single unique item. Consider an oracle which evaluates sets as follows. For any combination of sets the oracle evaluates the cardinality of the union of the subsets exactly, except for a few special cases: For $S = A \cup b \forall A \subseteq A, b \in B$ the oracle returns $\tilde{f}(S) = 2$, and for $S \subseteq A$ the oracle returns $\tilde{f}(S) = 2 + \delta$ for some arbitrarily small $\delta > 0$. With access to this oracle, the greedy algorithm will only select sets in $A$ which may be as bad as linear in the size of the input.

The fact that the greedy algorithm fails does not yet exclude the possibility that a variant of greedy or perhaps some other algorithm can overcome small errors. To discuss this rigorously we need a formal model of optimization that accounts for error and noise.

1.1 Separation between error and noise

For a given function $f : 2^N \to \mathbb{R}$ and some $\epsilon > 0$ we say that an oracle $\tilde{f} : 2^N \to \mathbb{R}$ is $\epsilon$-erroneous if for every set $S \subseteq N$, it respects:

$$(1 - \epsilon)f(S) \leq \tilde{f}(S) \leq (1 + \epsilon)f(S)$$

Stated in these terms, in the max-coverage example we tricked the greedy algorithm with a 1/3-error oracle. The same consequences apply to an $\epsilon$-erroneous oracle for any $\epsilon > 0$ by planting $1/\epsilon$ items in $A$. Notice that given access to an $\epsilon$-erroneous oracle, one can trivially approximate the solution well in exponential time by evaluating all possible subsets and return the best solution. Is there a polynomial time algorithm that can obtain desirable approximation guarantees for maximizing a monotone submodular function under a cardinality constraint with access to $\epsilon$-error oracles? We exclude this possibility by proving the following theorem in Section 6.

**Theorem.** No randomized algorithm can obtain an approximation strictly better than $O(n^{-1/2+\delta})$ to maximizing monotone submodular functions under a cardinality constraint using $e^{n\delta}/n$ queries to an $\epsilon$-erroneous oracle, for any fixed $\epsilon, \delta < 1/2$, with high probability.

The above statement implies that even near-optimal approximations of submodular functions do not provide us with guarantees suitable for optimization. In particular, even constant-factor estimations of submodular functions from learning and sketching do not suffice.

**Optimization under noise.** Note that we can equivalently say that $\tilde{f} : 2^N \to \mathbb{R}$ is $\epsilon$-erroneous if for every $S \subseteq N$ we have that $\tilde{f}(S) = \xi_S f(S)$ for some $\xi_S \in [1 - \epsilon, 1 + \epsilon]$. The lower bound stated above applies to the case in which the error multipliers $\xi_S$ are adversarially chosen. A natural question is whether some relaxation of the adversarial error model can lead to possibility results.

**Definition.** For a function $f : 2^N \to \mathbb{R}$ we say that $\tilde{f} : 2^N \to \mathbb{R}$ is a **noisy** oracle if there exists some distribution $D$ s.t. $\tilde{f}(S) = \xi_S f(S)$ where $\xi_S$ is independently drawn from $D$ for every $S \subseteq N$. 


Note that a noisy oracle as defined above is consistent: for any \( S \subseteq N \) the noisy oracle returns the same answer \( \xi_S \) regardless of how many times it is queried. When the noisy oracle is inconsistent, mild conditions on the noise distribution allow the noise to essentially vanish after polynomially-many queries, reducing the problem to standard submodular optimization (see e.g. [45]). Note also that in the definition we do not impose any restrictions on the distribution. In particular, it is not required that \( \xi_S \in [1 - \epsilon, 1 + \epsilon] \).

**Noise distributions.** Naturally, if the distribution always returns 0, any algorithm with access to a noisy oracle is helpless. We will therefore be interested in defining a broad class of distributions that avoids such trivialities, and is general enough to contain natural classes of distributions.

**Definition 1.1.** A noise distribution \( D \) has a generalized exponential tail if there exists some \( x_0 \) such that for \( x > x_0 \) the probability density function \( \rho_D(x) = e^{-g(x)} \), where \( g(x) = \sum_{i} a_i x^{\alpha_i} \). We do not assume that all the \( \alpha_i \)'s are integers, but only that \( \alpha_0 \geq \alpha_1 \geq \ldots \), and that \( \alpha_0 \geq 1 \).

For \( i > 0 \) it is possible that \( \alpha_i < 1 \). This implies that a generalized exponential tail also includes cases where the probability density function denoted \( \rho_D \) respects \( \rho_D(x) = \rho_D(x_0) e^{-g'(x-x_0)} \). This is because we can simply add \( \rho_D(x_0) \) to \( g \) using \( \alpha_i = 0 \) for some \( i \), and do a coordinate change moving from \( g'(x-x_0) \) to an equivalent \( g(x) \). Note that in particular if \( D \) is a Gaussian or an Exponential distribution it belongs to the family of generalized exponential tail distributions. Technically, the definition of a generalized exponential tail doesn’t allow a bounded distribution. However, all our results also hold for bounded distributions, assuming that if \( x_0 \) is the supremum of the distribution then \( \rho_D(x_0) \neq 0 \) which then includes the uniform distribution\(^1\).

An additional requirement that we have on the distribution is that it is bounded away from zero. In particular, we assume that the likelihood of obtaining values that are smaller than some polynomial in \( n \) is polynomially small. From here on, we will assume every distribution \( D \) has a generalized exponential tail bounded away from zero as defined above.

### 1.2 Main results

Our main result is that for the problem of optimizing a monotone submodular function under cardinality constraints, near-optimal approximations are achievable under noise.

**Theorem.** For any monotone submodular function, given access to a noisy oracle of a generalized exponential tail distribution, there is a polynomial-time algorithm for optimizing the function under a cardinality constraint whose approximation ratio is w.h.p arbitrarily close to \( 1 - 1/e \) when \( k > 2 \).

More specifically we show:

- **1 - 1/e - \epsilon guarantee for large** \( k \in \Omega(\log \log n/e^2) \): For \( k \) that is sufficiently larger than \( \log \log n \) we give an algorithm which obtains a \( (1 - 1/e - \epsilon) \) approximation guarantee.

- **1 - 1/e - \epsilon guarantee for** \( k \in \Omega(1/e) \): For values of \( k \) that are smaller than \( O(\log \log n) \) the problem is surprisingly harder. We develop a different algorithm for this case which achieves the coveted \( (1 - 1/e - \epsilon) \) guarantee.

- **1 - 1/k - \epsilon guarantee for any constant** \( k \geq 2 \): When \( k \) is a small constant, a variant of the algorithm for \( k \in \Omega(1/e) \) achieves a \( 1 - 1/k - \epsilon \) approximation. Note that this gives a \( 1 - 1/e - \epsilon \) for any \( k \geq 2 \). We also give a \( k/(k+1) \) approximation in expectation for any \( k \). For \( k = 1 \) this is tight as no randomized algorithm can obtain an approximation ratio better than \( 1/2 + O(1/\sqrt{n}) \) and \( 2k-1)/2k + O(1/\sqrt{n}) \) for general \( k \).

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\(^1\)In fact the results hold even if \( \rho_D(x_0) = 0 \), as long as the decay to 0 is slower then doubly exponential.
1.3 Overview of techniques

The celebrated greedy algorithm for submodular maximization discussed above crucially depends on the ability to perform a simple operation: find argmax_{a \in N} f(S \cup a). In the presence of noisy evaluations, we cannot evaluate sets exactly and the naive greedy algorithm becomes inapplicable. Of course, if we had a procedure that computes argmax_{a \in N} f(S \cup a) with sufficiently high probability, we could then apply the greedy algorithm with this procedure. Unfortunately, we do not know how to construct such a procedure with access to a noisy oracle. Instead, we design constructions of slightly weaker operators which, given a set S add an elements in a manner that may not be optimal, though still provide desirable guarantees. We generally refer to this technique as smoothing.

**Smoothing.** Intuitively, smoothing enables us to identify elements whose marginal contribution is large. To get some intuition behind this idea, consider evaluating a continuous function with access to a noisy oracle. One reasonable approach to obtain the value of a certain point is by sampling noisy valuations of polynomially-many points around it and use their average as an estimation for the true value. Intuitively, we mimic this approach by selecting a family of sets \( H \) and for every pair of sets \( A, B \subseteq N \) we average the values of \( \tilde{f}(A \cup H) \) and \( \tilde{f}(B \cup H) \) for all \( H \in H \), and select the set whose averaged noisy contribution is maximal. The smoothing arguments essentially make the noise disappear and instead leave us to deal with the implications of dealing with proxies of marginal contributions on optimization. In that sense, a large part of the technical challenge involves arguments about the properties of submodular functions, rather than dealing with noise.

1.4 The limitations of the model

Our results hold for the case in which the noise is drawn i.i.d. It seems like the i.i.d assumption is necessary as simple examples show that when the noise is correlated, or when using multiple noise distribution, no non-trivial solutions are possible. Another criticism is that the distribution generating the noise is quite permissive, potentially making two sets whose true values are close have dramatically different values under noise. It seems reasonable that in many applications this shouldn’t be the case. For such applications, one can consider bounded distributions and in particular distributions whose values are in \([1-\epsilon, 1+\epsilon]\). Even for such degenerate cases, the classic greedy algorithm fails and we are not aware of simpler approaches than those presented in this paper.

1.5 Paper organization

We begin by laying out definitions and proving concentration bounds of smoothing in Section 2. These bounds are later used in the analysis of the algorithms in subsequent chapters. In Section 3 we describe and analyze the SLICK-GREEDY which can be applied when \( k \in \Omega(\log \log n/\epsilon) \). In Section 4 we describe and analyze the SM-GREEDY which can essentially be applied on \( k \in O(\log \log n) \). We give an algorithm for \( k \in O(1/\epsilon) \) and information-theoretic lower bounds on constant \( k \) in Section 5. We conclude with a lower bound on adversarial noise in Section 6.

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\(^2\)We do note however, that when the distribution is bounded, the running time of the algorithms can be significantly improved, and in Section 4 the analysis can be simplified. We highlight this and give the proof for this special in Section 4 as well.

\(^3\)These bounds may be best understood in the context of the algorithms that use them. Our recommendation for the reader is therefore to read the definitions and explanations, but skip the proofs of the concentration bounds and return to them once they are used in the analysis of the algorithms.
1.6 More related work

**Submodular optimization.** Maximizing monotone submodular functions under cardinality and matroid constraints have received considerable interest. The seminal works of [63, 37] show that the greedy algorithm gives a factor of $1 - 1/e$ for maximizing a submodular function under a cardinality constraint and a factor $1/2$ approximation for matroid constraints. For max-cover which is a special case of maximizing a submodular function under a cardinality constraint, Feige shows that no poly-time algorithm can obtain an approximation better than $1-1/e$ unless P=NP [28]. Vondrak presented the continuous greedy algorithm which gives a $1 - 1/e$ ratio for maximizing a monotone submodular function under matroid constraints [73]. This is optimal, also in the value oracle model [62, 47, 64]. It is interesting to note that with a demand oracle the approximation ratio is strictly better than $1 - 1/e$ [32]. When the function is not monotone, constant factor approximation algorithms are known to be obtainable as well [29, 56, 11, 12]. In general, in the past decade there has been a development in the theory of submodular optimization, through concave relaxations [1, 16], the multilinear relaxation [15, 73, 17], and general rounding technique frameworks [75]. In this paper, the techniques we develop arise from first principles: we only rely on basic properties of submodular functions, concentration bounds, and the algorithms are variants of the standard greedy algorithm.

**Submodular optimization in game theory.** Submodular functions have been studied in game theory almost fifty years ago [71]. In mechanism design submodular functions are used to model agents’ valuations [57] and have been extensively studied in the context of combinatorial auctions (e.g. [21, 22, 20, 62, 13, 19, 66, 26, 23]). Maximizing submodular functions under cardinality constraints have been studied in the context of combinatorial public projects [67, 70, 14, 61] where the focus is on showing the computational hardness associated with not knowing agents valuations and having to resort to incentive compatible algorithms. Our adversarial lower bound implies that if agents err in their valuations, optimization may be hard, regardless of incentive constraints.

**Submodular optimization in machine learning.** In the past decade submodular optimization has become a central tool in machine learning and data mining (see surveys [51, 52, 9]). Problems include identifying influencers in social networks [45, 69] sensor placement [58, 40], learning in data streams [72, 42, 54, 4], information summarization [59, 60], adaptive learning [41], vision [44, 43, 49], and general inference methods [50, 43, 18]. In many cases the submodular function is learned from data, and our work aims to address the case in which there is potential for noise in the model.

**Combinatorial optimization under noise.** Combinatorial optimization with noisy inputs can be largely studied through inconsistent (returns independent noisy answers when queried multiple times) and consistent oracles. For inconsistent oracles, it usually suffices to repeat every query $O(\log n)$ times, and eliminate the noise. To the best of our knowledge, submodular optimization has been studied under noise only in instances where the oracle is inconsistent or equivalently small enough so that it does not affect the optimization [45, 53]. One line of work is studies methods for reducing the number of samples required for optimization (see e.g. [31, 8]), primarily for sorting and finding elements. On the other hand, if two identical queries to the oracle always yield the same result, the noise can not be averaged out so easily, and one needs to settle for approximate solutions, which has been studied in the context of tournaments and rankings [46, 10, 2].

**Convex optimization under noise.** Maximizing functions under noise is also an important topic in convex optimization. The analogue of our model here is one where there is a zeroth-order noisy oracle to a convex function. As discussed in the paper, the question of polynomial-time algorithms for noisy convex optimization is straightforward and the work in this area largely aims at improving the convergence rate [27, 38, 48, 55, 68].
2 Combinatorial Smoothing

In this section we illustrate a general framework we call *combinatorial smoothing* that we will use in the subsequent sections. Intuitively, combinatorial smoothing mitigates the effects of noise and enables finding elements whose marginal contribution is high.

**Some intuition.** Recall from our discussion in Section 1.3 that implementing the greedy algorithm requires identifying \( \text{arg max} \ f(S \cup a) \) for a given set \( S \) of elements selected by the algorithm in previous iterations. Thus, if for some \( a, b \in N \) we can compare \( A = S \cup a \) and \( B = S \cup b \) and decide whether \( f(A) > f(B) \) or vice versa, we can implement the greedy algorithm. Put differently, viewing a set as a point on the hypercube, given two points in \( \{0, 1\}^n \) we need to be able to tell which one has the larger true value, using a noisy oracle. In a world of continuous optimization, a reasonable approach to estimate the true value of a point in \([0, 1]^n\) with access to a noisy oracle is to take a small neighborhood around the point, sample values of points in its neighborhood, and average their values. Taking polynomially-many samples allows concentration bounds to kick in, and using a small enough diameter can often guarantee that the averaged value is a reasonable estimate of the point’s true value. Surprisingly, the spirit of this simple idea can used in submodular optimization as well.

**Smoothing neighborhood.** For a given subset \( A \subseteq N \) a *smoothing function* is a method which assigns a family of sets \( \mathcal{H}(A) \) called the *smoothing neighborhood*. The smoothing function will be used to create a smoothing neighborhood for a small set \( A \). This set \( A \) whose marginal contribution we aim to evaluate, is essentially a candidate for a greedy algorithm. In the application in Section 3 the set \( A \) will simply be a single element, whereas in Section 4 the set \( A \) will be of size \( O(1/\epsilon) \).

**Definition 2.1.** For a given function \( f : 2^N \to \mathbb{R}, A, S \subseteq N \), and smoothing neighborhood \( \mathcal{H}(A) \):

- \( F_S(A) := \mathbb{E}_{X \in \mathcal{H}(A)} f_S(X) \) (the smooth marginal contribution of \( A \)),
- \( F(A; S) := \mathbb{E}_{X \in \mathcal{H}(A)} f(S \cup X) \) (the smooth value of \( S \cup A \))
- \( \tilde{F}(A; S) := \mathbb{E}_{X \in \mathcal{H}(A)} f(S \cup X) \) (the noisy smooth value of \( S \cup A \)).

The idea behind combinatorial smoothing is to select a smoothing neighborhood which includes sets whose value is in some sense close to the value of the set \( A \) whose marginal contribution we wish to evaluate. Intuitively, when the sets are indeed close, by averaging the values of the sets in \( \mathcal{H}(A) \) we can mitigate the effects of noise and produce meaningful statistics. For intuition see Figure 2.

**Bounding the effects of noise.** In the remainder of this section we will prove three lemmas that we use in subsequent chapters to bound the effects of noise. Intuitively, these lemmas imply that an element whose smooth contribution is high also has high smooth noisy contribution. Lemma 2.2 gives an upper bound on the value of a noise multiplier with exponentially high probability when the distribution has an exponentially decaying tail. Lemma 2.4 lower bounds the noisy smooth contribution of a set in terms of its (true) smooth contribution. Lemma 2.5 upper bounds the smooth noisy contribution of any element (not just relevant ones) against its smooth contribution. Together, these lemmas are used in subsequent sections to argue about properties of elements selected by the algorithm. In particular, they are used to show that when selecting the element with the largest noisy contribution, that element is an arbitrarily good approximation to the element with the largest smooth marginal contribution.
Figure 2: An illustration of smoothing. For every element in the ground set we associate an index $i \in [n]$ and define the submodular function as $f(S) = \sqrt{\sum_{i \in S} i/2} - c$ for a constant $c > 0$. The blue dot depicts the true value of the element $a$ associated with the index $i = 400$ and the red dot depicts the true value of the element $b$ associated with the index $j = 900$. The light blue and light red dots depict the noisy function values of elements associated with indices $i$ in the range $|i - 400| \leq 100$ and $|i - 900| \leq 100$. For $S = \emptyset$, and smoothing neighborhoods $\mathcal{H}(a) = \{i : |i - a| \leq 100\}$ and $\mathcal{H}(b) = \{i : |i - b| \leq 100\}$ we depict $\tilde{F}(a; S)$ and $\tilde{F}(b; S)$ as the blue and red triangles, respectively. Intuitively, an algorithm which needs to decide whether $a$ (blue point) is larger than $b$ (red point) will decide by comparing $\tilde{F}(a; S)$ (blue triangle) and $\tilde{F}(b; S)$ (red triangle).

Recommendation. At this point, the reader may benefit by skipping to the following section. The motivation for the three lemmas in this section is perhaps best understood in the context of the analysis of the algorithms that employ them. We leave them here as they their proof is a necessary preliminary to the arguments ahead, and they are used by sections 3, 4, and 5.

2.1 Smoothing arguments

Lemma 2.2. Let $\omega_{\text{max}}$ and $\omega_{\text{min}}$ be the upper and lower bounds on the value of the noise multiplier in any of the calls made by a polynomial-time algorithm. For any $\delta > 0$, we have that:

- $\Pr[\omega_{\text{max}} < t^\delta] > 1 - e^{-\Omega(t^\delta/\ln t)}$
- $\Pr[\omega_{\text{min}} > t^{-\delta}] > 1 - e^{-\Omega(t^\delta/\ln t)}$

Proof. As $n$ tends to infinity, this is lemma trivial for any noise distribution which is bounded, or has finite support. If the noise distribution is unbounded, we know that its tail is subexponential. Thus, at any given sample the probability of seeing the value $n^\delta$ is at most $e^{-O(n^\delta)}$ where the
constant in the big $O$ notation depends on the magnitude of the tail. Iterating this a polynomial number of times gives the bound. The proof of the lower bound is equivalent.

\[ \lambda = \min_{T \in \mathcal{H}(A)} \frac{f(T)}{\omega} \]

**Lemma 2.4.** Let $f : 2^N \to \mathbb{R}$, $A, S \subseteq N$, $\omega = \max_{A_i \in \mathcal{H}(A)} \xi A_i$, and $\mu$ be the mean of the noise distribution. For $\epsilon = \min \{1, 2v_S(\mathcal{H}) \cdot |\mathcal{H}(A)|^{-1/4}\}$ with probability $1 - e^{-\Omega(\lambda^2 t/\omega)}$ we have:

\[ \tilde{F}(A; S) > (1 - \lambda) \mu \cdot (f(S) + (1 - \epsilon) \cdot F_S(A)). \]

**Proof.** Let $A_1, \ldots, A_t$ be the sets in $\mathcal{H}(A)$ and let $\alpha_1, \ldots, \alpha_t$ denote the corresponding marginal contributions and $\xi_1, \ldots, \xi_t$ denote their noise multipliers. In these terms the noisy smooth value is:

\[ \tilde{F}(A; S) = \frac{1}{t} \sum_{i=1}^{t} \xi_i (f(S) + \alpha_i) = \frac{1}{t} \sum_{i=1}^{t} \xi_i f(S) + \frac{1}{t} \sum_{i=1}^{t} \xi_i \alpha_i. \]

Let $\omega$ be the upper bound on the value of the noise multiplier. Applying the Chernoff bound, we get that for any $\lambda \in [0, 1]$ with probability at least $1 - e^{-\Omega(\lambda^2 t/\omega)}$:

\[ \frac{1}{t} \sum_{i=1}^{t} \xi_i f(S) \geq (1 - \lambda) \mu f(S) \]

To complete the proof we need to argue about concentration of the second term in (1). To do so, in our analysis we will consider a fine discretization of $\{\alpha_i\}_{i \in [t]}$ and apply concentration bounds on each discretized value. Define $\alpha_{\max} = \max_{i \in [t]} \alpha_i$ and $\alpha_{\min} = \min_{i \in [t]} \alpha_i$. We can divide the set of values $\{\alpha_i\}_{i \in [t]}$ to $t^{1/4}$ bins $\text{BIN}_1, \ldots, \text{BIN}_{t^{1/4}}$, where a value $\alpha_i$ is placed in the bin $\text{BIN}_q$ if

\[ (q - 1) \cdot \alpha_{\max} t^{-1/4} \leq \alpha_i \leq q \cdot \alpha_{\max} t^{-1/4} \]

Say a bin is dense if it contains at least $t^{1/4}$ values and sparse otherwise. Consider some dense bin $\text{BIN}_q$ and let $\alpha_{\min(q)} = \min_{i \in \text{BIN}_q} \alpha_i$ and $\alpha_{\max(q)} = \max_{i \in \text{BIN}_q} \alpha_i$. Since every bin is of width $\alpha_{\max} \cdot t^{-1/4}$ we know that:

\[ \alpha_{\min(q)} \geq \alpha_{\max(q)} - \alpha_{\max} \cdot t^{-1/4} \]

Applying concentration bounds as above, we get that $\sum_{i \in \text{BIN}_q} \xi_i \geq (1 - \lambda) \mu \cdot |\text{BIN}_q|$ with probability at least $1 - e^{-\Omega(\lambda^2 t/\omega)}$ for any $\lambda \in [0, 1]$. Thus, with this probability:

\[ \sum_{i \in \text{BIN}_q} \xi_i \alpha_i \geq \sum_{i \in \text{BIN}_q} \xi_i \alpha_{\min(q)} \]

\[ \geq (1 - \lambda) \mu \cdot |\text{BIN}_q| \cdot \alpha_{\min(q)} \]

\[ \geq (1 - \lambda) \mu \cdot |\text{BIN}_q| \cdot \left( \max \left\{ 0, \alpha_{\max(q)} - \alpha_{\max} \cdot t^{-1/4} \right\} \right) \alpha_{\max(q)} \]

\[ \geq (1 - \lambda) \mu \cdot |\text{BIN}_q| \cdot \left( \max \left\{ 0, 1 - \frac{\alpha_{\max}}{\alpha_{\max(q)}} \cdot t^{-1/4} \right\} \right) \alpha_{\max(q)} \]

\[ = (1 - \lambda) \mu \cdot |\text{BIN}_q| \cdot \left( \max \left\{ 0, 1 - v_S(\mathcal{H}(A)) \cdot t^{-1/4} \right\} \right) \alpha_{\max(q)} \]

\[ \geq (1 - \lambda) \mu \cdot |\text{BIN}_q| \cdot \left( \max \left\{ 0, 1 - v_S(\mathcal{H}(A)) \cdot t^{-1/4} \right\} \right) \alpha_{\max(q)} \]
Taking a union bound over all (at most $t^{1/4}$) dense bins, we get that with probability $1-e^{-\Omega(\lambda^2 t^{1/4}/\omega)}$:

$$
\sum_{i \in \text{dense}} \xi_i \alpha_i \geq (1 - \lambda) \mu \cdot \left( 1 - \max \left\{ 0, \nu_S(\mathcal{H}(A)) \cdot t^{-1/4} \right\} \right) \sum_{i \in \text{dense}} |\text{BIN}_q| \cdot \alpha_{\max(q)} \\
\geq (1 - \lambda) \mu \cdot \left( \max \left\{ 0, 1 - \nu_S(\mathcal{H}(A)) \cdot t^{-1/4} \right\} \right) \sum_{i \in \text{dense}} \alpha_i. \tag{2}
$$

Let $\alpha = \frac{1}{t} \sum_{i=1}^t \alpha_i$. Since we have less than $t^{1/4}$ elements in a sparse bin, and in total $t^{1/4}$ bins, the number of elements in sparse bins is at most $t^{1/2}$. We can use this to effectively lower bound the values in sparse bins in terms of $\alpha$:

$$
\sum_{i \in \text{dense}} \alpha_i = \sum_{i=1}^t \alpha_i - \sum_{i \in \text{sparse}} \alpha_i \\
\geq \max \left\{ 0, \sum_{i=1}^t \alpha_i - t^{1/2} \alpha_{\max} \right\} \\
\geq \max \left\{ 0, t \alpha - t^{1/2} \alpha_{\max} \right\} \\
> \max \left\{ 0, t \cdot \left( 1 - \frac{\alpha_{\max}}{\alpha_{\min}} \cdot t^{-1/2} \right) \alpha \right\} \\
= \max \left\{ 0, t \cdot \left( 1 - \nu_S(\mathcal{H}) \cdot t^{-1/2} \right) \alpha \right\} \tag{3}
$$

Putting (2) and (3) we get that for any $\lambda < 1$, with probability $1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)}$:

$$
\tilde{F}_S(A) = \frac{1}{t} \sum_{i=1}^t \xi_i \cdot \alpha_i \\
\geq \frac{1}{t} \sum_{i \in \text{dense}} \xi_i \cdot \alpha_i \\
\geq (1 - \lambda) \mu \cdot \left( \max \left\{ 0, 1 - \nu_S(\mathcal{H}(A)) \cdot t^{-1/4} \right\} \right) \cdot \frac{1}{t} \sum_{i \in \text{dense}} \alpha_i \\
\geq (1 - \lambda) \mu \cdot \left( \max \left\{ 0, 1 - \nu_S(\mathcal{H}(A)) \cdot t^{-1/4} \right\} \right) \left( \max \left\{ 0, 1 - \nu_S(\mathcal{H}(A)) \cdot t^{-1/2} \right\} \right) \alpha \\
> (1 - \lambda) \mu \cdot \left( \max \left\{ 0, 1 - 2 \nu_S(\mathcal{H}(A)) \cdot t^{-1/4} \right\} \right) \alpha \\
= (1 - \lambda) \mu \cdot \left( \max \left\{ 0, 1 - 2 \nu_S(\mathcal{H}(A)) \cdot t^{-1/4} \right\} \right) F_S(A)
$$

Taking a union bound we get that for any positive $\lambda < 1$ with probability $1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)}$:

$$
\tilde{F}(A; S) = \frac{1}{t} \sum_{i=1}^t \xi_i f(S) + \frac{1}{t} \sum_{i=1}^t \xi_i \alpha_i \\
> (1 - \lambda) \mu \cdot \left( f(S) + \max \left\{ 0, 1 - 2 \nu_S(\mathcal{H}(A)) \cdot t^{-1/4} \right\} \cdot F_S(A) \right) \\
= (1 - \lambda) \mu \cdot \left( f(S) + \left( 1 - \min \left\{ 1, 2 \nu_S(\mathcal{H}(A)) \cdot t^{-1/4} \right\} \cdot F_S(A) \right) \right). \\
\square
$$

Lemma 2.5. Let $f : 2^N \to \mathbb{R}, A, S \subseteq N, \omega = \max_{A_i \in \mathcal{H}(A)} \xi_{A_i}, \alpha_{\max} = \max_{A_i \in \mathcal{H}(A)} f_S(A_i)$ and $\mu$ be the mean of the noise distribution. For $\epsilon = 3 t^{-1/4} \alpha_{\max}$ we have that for any $\lambda < 1$ with probability $1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)}$:

$$
\tilde{F}(A; S) < (1 + \lambda) \mu \cdot (f(S) + F_S(A) + \epsilon).
$$
Proof. As in the proof of Lemma 2.4 let \( A_1, \ldots, A_t \) denote the sets in \( \mathcal{H}(A) \), and for each set \( A_i \) we will again use \( \alpha_i \) to denote the marginal value \( f_\mathcal{S}(A_i) \) and \( \xi_i \) to denote the noise multiplier \( \xi_{S \cup \{A_i\}} \).

As before, we will focus on showing concentration on the second term. Define \( \alpha_{\text{max}} = \max_i \alpha_i \) and \( \alpha_{\text{min}} = \min_i \alpha_i \). To apply concentration bounds on the second term, we again partition the values of \( \{\alpha_i\}_{i \in [t]} \) to bins of width \( \alpha_{\text{max}} \cdot t^{-1/4} \) and call a bin dense if it has at least \( t^{1/4} \) values and sparse otherwise. Using this terminology:

\[
\sum_{i=1}^{t} \xi_i \alpha_i = \sum_{i \in \text{dense}} \xi_i \alpha_i + \sum_{i \in \text{sparse}} \xi_i \alpha_i.
\]

Let \( \text{BIN}_d \) be the dense bin whose elements have the largest values. Consider the \( t^{1/4}/2 \) largest values in \( \text{BIN}_d \) and call the set of indices associated with these values \( L \). We have:

\[
\sum_{i=1}^{t} \xi_i \alpha_i = \sum_{i \in \text{dense} \setminus L} \xi_i \alpha_i + \sum_{i \in L \cup \text{sparse}} \xi_i \alpha_i.
\]

The set \( L \cup \text{sparse} \) is of size at least \( t^{1/4}/2 \) and at most \( t^{1/4}/2 + t^{1/2} \). This is because \( L \) is of size exactly \( t^{1/4}/2 \) and there are at most \( t^{1/2} \) values in bins that are sparse since there are \( t^{1/4} \) bins and a bin that has at least \( t^{1/4} \) is already considered dense. Thus, when \( \omega \) is an upper bound on the value of the noise multiplier, from Chernoff, for any \( \lambda \in [0, 1] \) with probability \( 1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)} \):

\[
\sum_{i \in L \cup \text{sparse}} \xi_i \alpha_i \leq \sum_{i \in L \cup \text{sparse}} \xi_i \alpha_{\text{max}} < (1 + \lambda) \mu \cdot |L \cup \text{sparse}| \cdot \alpha_{\text{max}} \\
\leq (1 + \lambda) \mu \cdot \left( \frac{t^{1/4}}{2} + t^{1/2} \right) \alpha_{\text{max}} \\
< (1 + \lambda) \mu \cdot 2t^{1/2} \alpha_{\text{max}}
\]

We will now use the same logic as in the proof of Lemma 2.4 to apply concentration bounds on the values in the dense bins. For a dense bin \( \text{BIN}_q \), let \( \alpha_{\text{max}(q)} \) and \( \alpha_{\text{min}(q)} \) be the maximal and minimal values in the bin, respectively. As in Lemma 2.4, for any \( \lambda \in [0, 1] \) with probability \( 1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)} \):

\[
\sum_{i \in \text{BIN}_q} \xi_i \alpha_i \leq \sum_{i \in \text{BIN}_q} \xi_i \cdot \alpha_{\text{max}(q)} \\
\leq (1 + \lambda) \mu \cdot \alpha_{\text{max}(q)} \cdot |\text{BIN}_q| \\
\leq (1 + \lambda) \mu \cdot \left( \alpha_{\text{min}(q)} + \alpha_{\text{max}} \cdot t^{-1/4} \right) \cdot |\text{BIN}_q| \\
< (1 + \lambda) \mu \cdot \left( |\text{BIN}_q| \cdot \alpha_{\text{min}(q)} + |\text{BIN}_q| \alpha_{\text{max}} \cdot t^{-1/4} \right)
\]

Applying a union bound we get with probability \( 1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)} \):

\[
\sum_{i \in \text{dense} \setminus L} \xi_i \alpha_i < \sum_{q} (1 + \lambda) \mu \cdot \left( |\text{BIN}_q| \cdot \alpha_{\text{min}(q)} + |\text{BIN}_q| \alpha_{\text{max}} \cdot t^{-1/4} \right) \\
< (1 + \lambda) \mu \cdot t \left( \alpha + t^{-1/4} \alpha_{\text{max}} \right)
\]
Together we have:

\[
\frac{1}{t} \sum_{i=1}^{t} \xi_i \alpha_i = \frac{1}{t} \left( \sum_{i \in \text{dense}\backslash L} \xi_i \alpha_i + \sum_{i \in L \backslash \text{sparse}} \xi_i \alpha_i \right) < (1 + \lambda) \mu \cdot \left( \alpha + t^{-1/4} \alpha_{\max} + 2t^{-1/2} \alpha_{\max} \right) \\
< (1 + \lambda) \mu \cdot \left( \alpha + 3t^{-1/4} \alpha_{\max} \right) \\
< (1 + \lambda) \mu \cdot \left( F_S(A) + 3t^{-1/4} \alpha_{\max} \right)
\]

By a union bound we get that with probability \(1 - e^{-\Omega(t^{2/4}/\omega)}\):

\[
\tilde{F}(A; S) = \frac{1}{t} \sum_{i=1}^{t} \xi_i f(S) + \frac{1}{t} \sum_{i=1}^{t} \xi_i \alpha_i \leq (1 + \lambda) \mu \cdot \left( f(S) + F_S(A) + 3t^{-1/4} \alpha_{\max} \right).
\]

### 3 Optimization for Large \(k\)

In this section we describe a deterministic algorithm that can be applied when the cardinality constraint \(k\) is sufficiently large, i.e. \(k \in \Omega(\log \log n)\). In particular, given a desired degree of accuracy \(\epsilon > 0\), our guarantees apply when \(k \geq 4608 \log \log n/\epsilon^2\). We will first describe the smoothing neighborhood which intuitively helps us find an element with sufficiently large marginal contribution. We then describe the \textsc{Smooth-Greedy} algorithm that our main algorithm uses as a subroutine, and analyze its performance. We conclude with the main result of this section which is a description and analysis of the \textsc{Slick-Greedy} algorithm whose approximation is arbitrarily close to \(1 - 1/e\).

#### 3.1 The smoothing neighborhood

The operator we construct to compare between two sets \(A, B \subseteq N\) can be described as follows. We select an arbitrary set \(H\) and for a given element \(a\), the smoothing neighborhood is simply \(\mathcal{H} = \{H' \cup a : H' \subseteq H\}\). Throughout the rest of this section we assume that \(H\) is an arbitrary subset of size \(\ell\), where the size of \(\ell\) depends on \(k\). In the case where \(k \geq 1152 \log n\) we will assign \(\ell = 12 \log n\), and when \(k < 1152 \log n\) we will have \(\ell = 48 \log \log n\). \(^4\) We will use \(t = 2^\ell\) to denote the number of subsets of \(H\) and \(k' = k - \ell\). The precise choice for \(\ell\) will become clear later in this section. Intuitively, \(\ell\) is on the one hand small enough so that if \(k\) is sufficiently large we can afford to sacrifice \(\ell\) elements for smoothing the noise, and on the other hand \(\ell\) is large enough so that taking all its subsets gives us a large smoothing neighborhood which enables applying strong concentration bounds. Denoting \(H^1, \ldots, H^t\) as all the subsets of \(H\), the smooth contribution and smooth marginal contribution are:

\[
F(a; S) = \frac{1}{t} \sum_{i=1}^{t} f(S \cup H^i \cup a)
\]

\[
F_{S}(a) = \frac{1}{t} \sum_{i=1}^{t} f_{S}(H^i \cup a)
\]

\(^4\)W.l.o.g. we assume that \(k < n - 4 \log n\) as for sufficiently large \(n\) this then implies that \(k \geq (1 - \epsilon)n\) and by submodularity optimizing with \(k' = n - 4 \log n\) suffices to get the \(1 - 1/e - \epsilon\) guarantee.
3.2 The smooth greedy algorithm

The smooth greedy algorithm is a variant of the standard greedy algorithm which replaces the procedure of finding $\arg \max_{a \in N} f(S \cup a)$ with its smooth analogue. The algorithm receives an arbitrary set of elements $H$ of size $\ell$ and at every stage adds the element $a \in N \setminus (S \cup H)$ to the solution $S$ for which the smooth noisy contribution is largest.

Algorithm 1 Smooth-Greedy

Input: budget $k$, set $H$
1: $S \leftarrow \emptyset$
2: while $|S| < k - |H|$ do
3: \[ S \leftarrow S \cup \arg \max_{a \in N \setminus (S \cup H)} \tilde{F}(a; S) \]
4: end while
5: return $S$

At a high level, the idea behind the analysis is to compare the performance of the solution returned by the algorithm against the optimal solution which ignores $H$. That is, we compare against the optimal solution evaluated on $f_H$ where $f_H(S) = f(H \cup S) - f(H)$. To gain some intuition behind this idea in the analysis we encourage the reader to consult Figure 3. Essentially, we will show that at every step Smooth-Greedy selects an element whose marginal contribution is larger than that of the optimal solution evaluated on $f_H$. We will use $\OPT$ and $\OPT_T$ to denote the value optimal solution using $k'$ elements on $f$ and $f_H$, respectively.

Definition 3.1. Let $O = \arg \max_{T: |T| \leq k'} f(T)$ and $S$ the current set of elements in some iteration of Smooth-Greedy. The iteration is $\epsilon$-relevant if $f_{H \cup \overline{S}}(O) \geq \epsilon \cdot \OPT_T$.

We will analyze Smooth-Greedy in the case where the iterations are $\epsilon$-relevant as it allows applying the smoothing arguments. For iterations that are not $\epsilon$-relevant, we will later see that we can ignore these iterations in our analysis and the loss in the approximation guarantee is negligible.

The main steps in the analysis of Smooth-Greedy are:

1. In Lemma 3.4 we show that when the iterations are $\epsilon$-relevant, the element selected in each iteration by the algorithm is with high probability an arbitrarily good approximation to the (non-noisy) smooth marginal contribution $F_S(a)$. To do so we need claims 3.2 and 3.3.

2. Next, in Claim 3.5 we show that the element $a$ whose smooth marginal contribution $F_S(a)$ is maximal has true marginal contribution $f_S(a)$ that is roughly a $k'$th fraction of the marginal contribution of the optimal solution over $f_H$.

3. Finally, in Lemma 3.6 we apply a standard inductive argument to show that the fact the algorithm selects an element with large marginal contribution in each step results in an approximation arbitrarily close to $1 - 1/e$ to $\OPT_T$ (not $\OPT$). In Corollary 3.7 we show that the bound against $\OPT_T$ can already be used to give a constant factor approximation to $\OPT$. To get arbitrarily close to $1 - 1/e$ we boost Smooth-Greedy as described in Section 3.3.

3.2.1 Smoothing guarantees

As this stage our goal is to prove Lemma 3.4. This lemma shows that at every step as Smooth-Greedy adds the element that maximizes the noisy contribution $\arg \max_{a \in N \setminus (S \cup H)} \tilde{F}(a; S)$, that element nearly maximizes the (non-noisy) smooth marginal contribution $F_S$, with high probability. We use claims 3.2 and 3.3 with lemmas 2.4 and 2.2 in the proof of Lemma 3.4.

Claim 3.2. For any $\epsilon > 0$ if $F_S(a) \geq (1 - \epsilon)F_S(b)$ then $f_S(a) \geq (1 - \epsilon)f_{S \cup H}(b)$.
Proof. Assume for purpose of contradiction that \( f_S(a) < (1 - \epsilon)f_{S \cup H}(b) \). Since \( f \) is a submodular function, the function \( f_S(T) = f(S \cup T) - f(S) \) is also submodular (and hence also subadditive). Notice that for any \( H' \subseteq H \), from subadditivity we have that \( f_S(H' \cup a) \leq f_S(a) + f_S(H') \) and from submodularity \( f_H(b) < f_H(b) \). Therefore, for any \( H' \subseteq H \) we get:

\[
\begin{align*}
    f_S(H' \cup a) & \leq f_S(H') + f_S(a) \\
    & < f_S(H') + (1 - \epsilon)f_{S \cup H}(b) \\
    & \leq f_S(H') + (1 - \epsilon)f_{S \cup H'}(b) \\
    &= (1 - \epsilon)f_S(H' \cup b).
\end{align*}
\]

Notice however, that this contradicts our assumption:

\[
F_S(a) = \frac{1}{k} \sum_{H' \subseteq H} f_S(H' \cup a) < (1 - \epsilon)\frac{1}{k} \sum_{H' \subseteq H} f_S(H' \cup b) = (1 - \epsilon)F_S(b).
\]

Claim 3.3. For an \( \epsilon \)-relevant iteration of Smooth-Greedy, let \( S \) be the set of elements selected in previous iterations, and \( \text{OPT}_H \geq \text{OPT}/3 \). If \( a^* \in \arg \max_{a \in A \setminus (S \cup H)} F_S(a) \) then \( v_S(H(a^*)) \leq 3k/\epsilon \).

Proof. Let \( O \in \arg \max_{T:|T| \leq k'} f_H(T) \), and let \( o^* \in \arg \max_{o \in O} f_{H \cup S}(o) \). By the maximality of \( a^* \) we have that \( F_S(a^*) \geq F_S(o^*) \), and thus by Claim 3.2 we get \( f_S(a^*) \geq f_{S \cup H}(a^*) \). Since the iteration is \( \epsilon \)-relevant we have that \( f_{S \cup H}(O) \geq \epsilon \cdot \text{OPT}_H \) and from subadditivity we get:

\[
F_{S \cup H}(o^*) \geq \frac{\epsilon \cdot \text{OPT}_H}{k'} \geq \frac{\epsilon \cdot \text{OPT}_H}{k}
\]

From \( f_S(a^*) \geq f_{S \cup H}(o^*) \) and \( f_{S \cup H}(o^*) \geq \epsilon \cdot \text{OPT}_H/k \), and monotonicity of \( f \) we get:

\[
\min_{H' \subseteq H} f_S(H' \cup a^*) \geq f_S(a^*) \geq \epsilon \cdot \text{OPT}_H/k
\]

and since every set in \( H(a^*) \) is of size at most \( k \) we know that \( \max_{H' \subseteq H} f_S(H' \cup a^*) \leq \text{OPT} \). Together with the fact that \( \text{OPT} \leq 3\text{OPT}_H \) we get:

\[
v_S(H(a^*)) = \max_{H' \subseteq H} f_S(H' \cup a^*) \leq \frac{\text{OPT}}{\epsilon} \cdot \frac{k}{\text{OPT}_H} \leq \frac{3k}{\epsilon}.
\]

We can now show (assuming \( \text{OPT}_H \) is sufficiently large) that in \( \epsilon \)-relevant iterations the value of the element which maximizes the noisy smooth value is comparable to that of the (non-noisy) smooth value, with exponentially high probability.
Lemma 3.4. For a fixed $\epsilon > 0$ and an $\epsilon$-relevant iteration of Smooth-Greedy, let $S$ be the set of elements selected in previous iterations, and assume $OPT_H \geq OPT/3$. Let $a \in \arg\max_{b \in N \setminus (S \cup H)} F(b; S)$ be the element selected at that stage. Then, w.p. at least $1 - 1/n^4$:

$$F_S(a) \geq (1 - \epsilon) \max_{b \in N \setminus (S \cup H)} F_S(b).$$

Proof. Let $a^* \in \arg\max_{a \in N \setminus (S \cup H)} F_S(a)$. We will show that for any $b \in N \setminus S \cup H$ for which $F_S(b) < (1 - \epsilon) F_S(a^*)$ we get that $\tilde{F}(b; S) < \tilde{F}(a^*; S)$ with high probability by lower bounding $F(a^*; S)$ and upper bounding $\tilde{F}(b; S)$ when $\omega$ (the upper bound on the largest realized noise multiplier) is bounded by some polynomial in $t$. Thus, by taking a union bound and conditioning on the event that $\omega$ is bounded (Lemma 2.2) we can conclude that after comparing against all elements in $N \setminus (S \cup H)$ an element whose smooth marginal contribution is at least $(1 - \epsilon)$ of that of $a^*$ will surface with exponentially high probability.

- **Lower bound on $F(a^*; S)$**: First, from Claim 3.3 we know that $v_S(\mathcal{H}(a^*)) \leq 3k/\epsilon$. Together with Lemma 2.4 we get with probability $1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)}$:

$$\tilde{F}(a^*; S) > (1 - \lambda)\mu \cdot \left( f(S) + \left( 1 - \frac{6k}{\epsilon} \cdot t^{-1/4} \right) \cdot F_S(a^*) \right)$$

- **Upper bound on $F(b; S)$**: Letting $\beta_{\text{max}}$ denote the set with the highest marginal contribution in $\mathcal{H}(b)$, from Lemma 2.5, we get that with probability $1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)}$:

$$\tilde{F}(b; S) < (1 + \lambda)\mu \cdot \left( f(S) + F_S(b) + 3t^{-1/4} \beta_{\text{max}} \right)$$

(5)

We’ll express this inequality in terms of $f(S)$ and $F_S(a^*)$ as well. First, notice that we assume $F_S(b) \leq (1 - \epsilon) F_S(a^*)$. Second, since all sets in $\mathcal{H}(b)$ are of size at most $k$ we also know that $\beta_{\text{max}} \leq OPT$. Since the iteration is $\epsilon$-relevant and $OPT_H \geq OPT/3$ this implies that

$$3t^{-1/4} \beta_{\text{max}} \leq 3t^{-1/4} \cdot OPT \leq 9t^{-1/4} \cdot k F_S(a^*) / \epsilon$$

To justify the last inequality, first notice that every set in $\mathcal{H}(a^*)$ includes $a^*$ we get that $F_S(a^*) \geq f_S(a^*)$. As in the arguments of the proof of Claim 3.3 due to the maximality of $a^*$ and Claim 3.2 we know that $f_S(a^*) \geq f_{S \cup H}(o^*)$ where $o^* \in \arg\max_{o \in O_H} F_S(o)$ and $O_H = \arg\max_{T \cup H \leq k} f_H(T)$. From $\epsilon$-relevant iteration and $OPT_H \geq OPT/3$ we get:

$$F_S(a^*) \geq f_S(a^*) \geq f_{S \cup H}(o^*) \geq f_{S \cup H}(O_H)/k \geq \epsilon \cdot OPT_H / k \geq \epsilon \cdot OPT / 3k$$

(6)

Therefore:

$$\tilde{F}(b; S) < (1 + \lambda)\mu \cdot \left( f(S) + \left( 9t^{-1/4} \cdot \frac{k}{\epsilon} + (1 - \epsilon) \right) F_S(a^*) \right)$$

(7)

Putting (5) together with (7) we get that with probability at least $1 - 2e^{-\Omega(\lambda^2 t^{1/4}/\omega)}$:

$$\tilde{F}(a^*; S) - \tilde{F}(b; S) > \mu \cdot \left( F_S(a^*) \left[ (1 - \lambda) \left( 1 - \frac{6k}{\epsilon} t^{-1/4} \right) - (1 + \lambda) \left( \frac{9k}{\epsilon} t^{-1/4} - (1 - \epsilon) \right) \right] - 2\lambda f(S) \right)$$

$$> \mu \cdot \left( F_S(a^*) \left[ \epsilon - \frac{15k}{\epsilon} t^{-1/4} - \lambda \left( 2 + \frac{3k}{\epsilon} t^{-1/4} \right) \right] - \lambda \frac{6k}{\epsilon} F_S(a^*) \right)$$

$$> \mu \cdot \left( F_S(a^*) \left[ \epsilon - \frac{k}{\epsilon} (15t^{-1/4} + 10\lambda) \right] \right)$$
The second inequality above is an application of 6 and the fact that \( f(S) < \OPT \) since \(|S| \leq k \).

The difference is strictly positive when \( t^{1/4} > 30k/\epsilon^2 \) and \( \lambda \leq \epsilon^2/20k \), and then holds with probability at least \( 1 - 2 \exp\left(-\Omega\left(\frac{\epsilon^4}{\omega k^2}\right)\right) \).

When \( k \geq 1152\log n \), we have that \( t = n^{12} \) and thus when \( n \geq 6/\epsilon \), we get \( \tilde{F}(a^*; S) - \tilde{F}(b; S) > 0 \) with probability at least \( 1 - 2 \exp\left(-\Omega(\epsilon^2 n/\omega)\right) \). Conditioning on the event in Lemma 2.2 with an upper bound on \( \omega = n^{1/3} \) we get the result. If \( k < 1152\log n \) then we have that \( t = \log^{18} n \). Since \( k < \log n \) the bound holds with probability at least \( 1 - \exp\left(-\Omega(\epsilon^4 \log^{10} n/\omega)\right) \). Conditioning on \( \omega < \log^5 n \) and employing Lemma 2.2 we get that the difference is strictly positive with probability at least \( 1 - \exp(-\Omega(\epsilon^4 \log^5 n)) \). For sufficiently large \( n \) we therefore get that \( F_S(a) - F_S(b) > 0 \) with probability at least \( 1 - 1/n^4 \) as required. \( \Box \)

### 3.2.2 Approximation guarantee

The above lemma releases us for a bit from thinking about noisy evaluations. Note that in cases where \( n \) is not sufficiently large, one can achieve these bounds by taking a larger polynomial in \( t \). We can now focus on analyzing the consequences of selecting an element \( a \) which (up to factor \( 1 - \epsilon \)) maximizes the smooth contribution \( F_S \) at every stage of the algorithm.

**Claim 3.5.** For any fixed \( \epsilon > 0 \), consider an \( \epsilon \)-relevant iteration of Smooth-Greedy, let \( S \) be the set of elements selected in previous iterations, and assume \( \OPT_H \geq \OPT / 3 \). Let \( a \in \arg \max_{b \in N \setminus (S \cup H)} \tilde{F}(b; S) \) be the element selected at that iteration. Then, with prob. \( > 1 - 1/n^4 \):

\[
F_S(a) \geq \left(1 - \epsilon\right) f_{H \cup S}(a^*)
\]

**Proof.** Let \( O \in \arg \max_{f_H|T| \leq k'} f_H(T) \), and \( o^* \in \arg \max_{o \in O} f_{H \cup S}(o) \). From Lemma 3.4 we know that with probability \( 1 - 1/n^5 \) we have \( F_S(a) \geq (1 - \epsilon) F_S(o^*) \). Claim 3.2 implies that:

\[
F_S(a) \geq (1 - \epsilon) f_{H \cup S}(o^*)
\]

From subadditivity \( f_{H \cup S}(o^*) \geq f_{H \cup S}(O)/k' \) and thus:

\[
F_S(a) \geq (1 - \epsilon) f_{H \cup S}(o^*) \geq \left(1 - \frac{\epsilon}{k'}\right) f_{H \cup S}(O) \geq \left(1 - \frac{\epsilon}{k'}\right) \left( f_H(O) - f(S) \right).
\]

**Lemma 3.6.** Let \( S \) be the set returned by Smooth-Greedy and \( H \) its smoothing set. Then, for any fixed \( \epsilon > 0 \) when \( k \geq 3\ell/\epsilon \) with probability of at least \( 1 - 1/n^3 \) we have that:

\[
f(S \cup H) \geq \left(1 - 1/e - \epsilon/3\right) \OPT_H.
\]

**Proof.** In case \( \OPT_H < \OPT / 3 \) then \( H \) alone provides a \( 1 - 1/e - \epsilon \) approximation. To see this, let \( O \in \arg \max_{T|T| \leq k} f(T) \) and \( O' \in \arg \max_{T|T| \leq k' \ell} f(T) \), and \( O_H \in \arg \max_{T|T| \leq k'} f_H(T) \). We get:

\[
(1 - \epsilon/3) f(O) \leq f(O') \tag{8}
\]

\[
t \leq f(H \cup O') \tag{9}
\]

\[
= f(H) + f_H(O') \tag{10}
\]

\[
\leq f(H) + f_H(O_H) \tag{11}
\]

\[
< f(H) + 1/3 f(O) \tag{12}
\]

The first inequality (8) is due to the fact that \( k' = k - \ell \) and \( k \geq 3\ell/\epsilon \); (9) is due to monotonicity, (11) is due to optimality of \( O_H \); (12) is due to the assumption that \( \OPT_H < 1/3OPT \). Thus:

\[
f(H) \geq (1 - 1/3 - \epsilon/3) f(O) \geq (1 - 1/e - \epsilon/3) \OPT \geq (1 - 1/e - \epsilon/3) \OPT_H
\]

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In case $\OPT_H \geq \OPT/3$ we can apply a standard induction argument on Claim 3.5 to show that $S$ alone provides a $1 - 1/e - \epsilon/3$ approximation. We will use the following notation. At every iteration $i \in [k']$ of the while loop in the algorithm, we will use $a_i$ to denote the element that was added in that step, and $S_i := \{a_1, \ldots, a_i\}$. For $\delta = \epsilon/6$, let $\hat{k} \leq k'$ be the last $\delta$-relevant iteration, i.e. the last iteration $i$ for which $f_{H \cup S_i}(O_H) \geq \delta \cdot \OPT_H$. For sufficiently large $n$ we can invoke Claim 3.5 with $\delta = \epsilon/2$ and take a union bound on all $k < n$ iterations. Let $\hat{O} \subseteq O_H$ be the subset of $\hat{k}$ elements whose value (in terms of $f_H$) is largest. Since our smoothing guarantees depend on the assumption that $f_{H \cup S_i}(O_H) \geq \delta \OPT_H$, we will analyze the algorithm against $\hat{O}$ whose total value is $(1 - \delta)\OPT_H$. Using this notation, Claim 3.5 shows that at every stage $i \in [\hat{k}]$:

$$f(S_{i+1}) \geq (1 - \delta) \left[ \frac{1}{\hat{k}} \left( f_H(\hat{O}) - f(S_i) \right) \right] + f(S_i).$$

We will show that at every stage $i \in [\hat{k}]$:

$$f(S_i) \geq (1 - \delta) \left( 1 - \left( 1 - \frac{1}{\hat{k}} \right)^i \right) f_H(\hat{O}).$$

The proof is by induction on $i$. For $i = 1$ we have that $S_i = \{a_1\}$ and invoking Claim 3.5 with $S = \emptyset$ we get that $f(a_i) \geq \frac{1}{\hat{k}} f_H(\hat{O})$. Therefore for $i = 1$:

$$f(S_1) = f(a_i) \geq (1 - \delta) \frac{1}{\hat{k}} f_H(\hat{O}) = (1 - \delta) \left( 1 - \left( 1 - \frac{1}{\hat{k}} \right) \right) f_H(\hat{O}).$$

We can now assume the claim holds for $i = l < \hat{k}$ and show that it holds for $i = l + 1$:

$$f(S_{l+1}) \geq (1 - \delta) \left( \frac{1}{\hat{k}} \left( f_H(\hat{O}) - f(S_l) \right) \right) + f(S_l)$$

$$\geq (1 - \delta) \left( \left( \frac{1}{\hat{k}} f_H(\hat{O}) \right) + \left( 1 - \frac{1}{\hat{k}} \right) f(S_l) \right)$$

$$\geq (1 - \delta) \left( \frac{1}{\hat{k}} f_H(\hat{O}) \right) + (1 - \delta) \left( 1 - \frac{1}{\hat{k}} \right) \left( 1 - \left( 1 - \frac{1}{\hat{k}} \right)^l \right) f_H(\hat{O})$$

$$= (1 - \delta) \left( 1 - \left( 1 - \frac{1}{\hat{k}} \right)^{l+1} \right) f_H(\hat{O})$$

Note that for any $l > 1$ we have that $(1 - 1/l)^l \leq 1/e$, and thus:

$$f(S) \geq f(S_{\hat{k}}) \geq (1 - 1/e - \delta) f_H(\hat{O}) \geq (1 - 1/e - 2\delta) \OPT_H = (1 - 1/e - \epsilon/3) \OPT_H.$$ 

Thus from monotonicity $f(S \cup H) \geq \max\{f(S), f(H)\} \geq (1 - 1/e - \epsilon/3) \OPT$. \hfill $\square$

Lemma 3.6 provides us the guarantee we need to analyze the algorithm in Section 3.3. It is worth noting that for $S$ returned by Smooth-Greedy and $H$ used for smoothing, the set $S \cup H$ is already a $(\frac{e-1}{2e-1} - \epsilon)$-approximation, for any $\epsilon > 0$ and sufficiently large $k$.

**Corollary 3.7.** Let $S$ be the set returned by Smooth-Greedy and $H$ be its smoothing set. For any fixed $\epsilon \in (0, 1/e]$ for which $k > |H|/\epsilon$, we have that with probability at least $1 - 1/n^3$:

$$f(S \cup H) > \left( \frac{e - 2}{2e - 2} - \epsilon \right) \OPT.$$
Proof. Let $O_H \in \arg\max_{T:|T| \leq k'} f_H(T)$. From Lemma 3.6, with probability at least $1 - 1/n$:

$$f(S \cup H) \geq (1 - 1/e - \epsilon)f(O_H)$$

Together with the assumption that $\epsilon \leq 1/e$ we get:

$$(1 - \epsilon)\text{OPT} \leq f(O_H \cup H) \leq f(O_H) + f(H) \leq \left(\frac{e}{e - 2}\right)f(S \cup H) + f(H)$$

and therefore $f(S \cup H) > \left(\frac{2e - 2}{2e - 2} - \epsilon\right)\text{OPT}$ as required.

3.3 The slick greedy algorithm: optimal approximation for sufficiently large $k$

We can now describe the Slick-Greedy algorithm which gives the main result of this section. The idea is to boost the smooth greedy algorithm. To do so, given a constant $\epsilon > 0$ we set $\delta = \epsilon/6$. We generate sets $H_1, \ldots, H_{1/\delta}$, each of size $\ell = 4 \log n$ s.t. $H_i \cap H_j = \emptyset$ for every $i, j \in [1/\delta]$. We then run the Smooth-Greedy algorithm $1/\delta$ times, where in each iteration $i$ we use $H_i$ to generate the smoothing neighborhood. For each such iteration we keep the solution returned by the algorithm $S_i$ and the set $H_i$. We then compare the solutions $T_i = S_i \cup H_i$ to the best set $T$ we’ve seen so far through a procedure we call Smooth-Compare described below. The Smooth-Compare procedure begins with a solution $T_i$ and compares it to another solution $T_j$ by generating a set $H_{ij}$ s.t. $H_{ij} \cap (T_j \cup T_i) = \emptyset$ and $|H_{ij}| = \ell$. If $T_i$ wins, the procedure returns $T_i$ and otherwise returns $T_j$. The Slick-Greedy then returns the set $T$ that survived the Smooth-Compare tournament.

Algorithm 2 Slick-Greedy

Input: budget $k$

1: Select $\ell/\delta$ elements in $N$ and partition them into disjoint sets of equal size $H_1, \ldots, H_{1/\delta}$
2: $T_i \leftarrow \emptyset$
3: for $j \in [1/\delta]$ do
4: $T_j \leftarrow$ Smooth-Greedy($k$, $H_j$) $\cup$ $H_j$
5: $H_{ij} \leftarrow$ arbitrary set of $\ell$ elements disjoint from $T_i \cup T_j$
6: $T_i \leftarrow$ Smooth-Compare($\{T_i, T_j\}, H_{ij}$)
7: end for
8: return $S$

The reason Smooth-Greedy alone cannot obtain an arbitrarily close approximation to $1 - 1/e$ is due to the fact that a substantial portion of the optimal solution’s value may be attributed to $H$. But this would be resolved if we had a way to guarantee that the contribution of the smoothing set we use is small. The idea behind the Slick-Greedy is do obtain this type of guarantee: by running a large albeit constant number of instances of Smooth-Greedy with different smoothing sets, and selecting the best solution we can ensure the contribution of the smoothing set is relatively minor.

Given a fixed precision $\epsilon > 0$ the main steps in the analysis are:

1. Lemma 3.8 shows that for at least one of the $H_i$’s we have that $\text{OPT}_{H_i} = \max_{T:|T| \leq k'} f_{H_i}(T)$ approximates $\text{OPT}$ up to a multiplicative factor of $(1 - \epsilon/3)$. This is a structural property of submodular functions, which is independent of the noise. Let $l$ be such an index.
2. We can then apply the guarantee of Lemma 3.6 on each $T_i$, which gives that with high probability $f(T_i)$ approximates $\OPT_{H_i}$ to a factor $(1 - 1/e - \epsilon/3)$.

3. We then prove that with high probability the set $T_i$ chosen by Slick-Greedy satisfies $f(T_i) \geq (1 - \epsilon/3)\max_{j \in [1/\delta]} f(T_j)$. In particular, $f(T_i) \geq (1 - \epsilon/3)f(T_i)$. This is shown in Corollary 3.9 which uses Claim 3.9 and Lemma 3.10. Putting everything together this implies that with high probability the set returned by Slick-Greedy is a $1 - 1/e - \epsilon$ approximation.

We now begin with proving the first step – showing that there is at least one set $H_i$ whose removal has little effect on the quality of the optimal solution.

**Lemma 3.8.** For any $\epsilon > 0$ and $\delta = \epsilon/6$ when $k > 36\ell/\epsilon^2$ then $\max_{i \in [1/\delta]} \OPT_{H_i} \geq (1 - \epsilon/3)\OPT$.

**Proof.** Let $O'$ be the optimal solution for $k' = k - \ell$ elements. If $k > 36\ell/\epsilon^2 = 6\ell\delta$, then since $f$ is submodular we have that $f(O') \geq (1 - \epsilon/6)\OPT$. From submodularity and since the $H_i$s are disjoint sets we get that:

$$f(O') \geq \sum_{i=1}^{1/\delta} f_{O\setminus H_i}(O' \cap H_i)$$

Since there are $1/\delta = 6/\epsilon$ elements in this sum, and the total sum is at most $f(O')$, there exists an index $l$ such that $f_{O\setminus H_i}(O' \cap H_i) \leq \epsilon f(O')/6$. In this case:

$$\OPT_{H_i} \geq f(O') - f_{O\setminus H_i}(O' \cap H_i) \geq (1 - \epsilon/6)f(O') \geq (1 - \epsilon/3)\OPT.$$

**The smooth comparison procedure.** We can now describe the SMOOTH-COMPARE procedure we use in the algorithm. For a given set $H_{ij} \subseteq N$ of size $\ell$ and two sets $T_i, T_j \subseteq N \setminus H_{ij}$, we compare $\tilde{f}(T_i \cup H_{ij})$ with $\tilde{f}(T_j \cup H_{ij})$ for all $H_{ij} \subseteq H_{ij}$. We select $T_i$ if in the majority of the comparisons with $H_{ij}' \subseteq H_{ij}$ (breaking ties lexicographically) we have that $\tilde{f}(T_i \cup H_{ij}') \geq \tilde{f}(T_j \cup H_{ij}')$, and otherwise we select $T_j$. A formal description is added below.

**Algorithm 3** SMOOTH-COMPARE

**Input:** $T_i, T_j, H_{ij} \subseteq N \setminus (T_i \cup T_j)$,

1: Compare $\tilde{f}(T_i \cup H_{ij}')$ with $f(T_j \cup H_{ij}')$ for all $H_{ij} \subseteq H_{ij}$

2: if $T_i$ won the majority of comparisons return $T_i$ otherwise return $T_j$

The third step from the high-level description of the analysis above is decomposed into two parts. First we show that when we compare two sets such that one is a factor of at least $1 - \epsilon\delta/3$ from the other (equivalently $(1 - \epsilon^2/18)$ from the other), the better set wins. Then we conclude that since we are running the SMOOTH-COMPARE tournament between $1/\delta$ sets, the winner is an $(1 - \epsilon\delta/3)^{1/\delta} \geq (1 - \epsilon/3)$ approximation to the competing set with the highest score.

**Claim 3.9.** Let $T_i = S_i \cup H_i$ and $T_j = S_j \cup H_i$ be two sets that are compared by SMOOTH-COMPARE, and suppose that $f(T_i) \geq (1 + 2\beta)f(T_j)$ where $\beta = |H_{ij}|/k'$. Then, for any set $H_{ij}' \subseteq H_{ij}$ with probability at least $1 - 1/n^3$:

$$f(T_i \cup H_{ij}') \geq f(T_j \cup H_{ij}').$$

**Proof.** Recall that $H_{ij} \cap (T_i \cup T_j) = \emptyset$. Let $H_{ij}' \subseteq H_{ij}$. Due to submodularity and the fact that every element in $H_{ij}'$ was a candidate for selection by the Slick-Greedy and wasn’t selected, this implies that the marginal contribution of $H_{ij}'$ to $S_i \cup H_i$ is bounded from above by $2\beta \cdot f(S_i)$ with exponentially high probability. To see this, let $a \in S_i$ be the last element chosen by the smooth greedy procedure, and let $H_{ij}' \subseteq H_{ij}$. Since $T_i \cap H_{ij} = \emptyset$ we know that every element $h \in H_{ij}' \subseteq H_{ij}$
was not chosen by the smooth greedy procedure since it is not in $S_i \subseteq T_i$, and from Claim 3.2 and Lemma 3.4 we know that with probability at least $1 - 1/n^4$:

$$f_{S_j \setminus \{a\}}(a) \geq (1 - \gamma) \max_{h \in H} f_{(S_j \setminus \{a\}) \cup H_j}(h)$$

From the above argument and submodularity, fixing $\gamma = 1/2$, applying a union bound over $O(\log n)$ elements of $H_{ij}$ we get that with probability at least $1 - 1/n^3$:

$$f_{T_j'(H'_{ij})} = f_{S_j \cup H_j}(H'_{ij})$$

$$\leq |H'_{ij}| \cdot \max_{h \in H} f_{S_j \cup H_j}(h)$$

$$\leq |H'_{ij}| \cdot \max_{h \in H} f_{(S_j \setminus \{a\}) \cup H_j}(h)$$

$$\leq (1 + \gamma)|H'_{ij}| \cdot f_{S_j \setminus \{a\}}(a)$$

$$\leq (1 + \gamma)\beta \cdot f(S_j)$$

$$< 2\beta \cdot f(S_j)$$

Notice that showing this suffices since then we get:

$$f(T_j \cup H'_{ij}) = f(T_j) + f_{T_j'(H'_{ij})} \leq (1 + 2\beta) f(T_j) < f(T_i) \leq f(T_i \cup H'_{ij})$$
Applying a Chernoff bound, for any constants $\epsilon, \delta > 0$, s.t. $\epsilon \delta / 8 > 1 + 2|H'_{ij}|/k'$, and $\nu_{\max} / \nu_{\min} \leq n^\tau$ for some constant $\tau > 0$, we get that $T_i$ is chosen with probability at least $1 - e^{-\Omega(n/\log(n))}$, conditioned on $\nu_{\max} / \nu_{\min} < n^\tau$ which by Lemma 2.2 occurs with probability $1 - e^{-\Omega(n^\alpha)}$ for some constant $\alpha > 0$. For sufficiently large $n$, $T_i$ therefore wins with probability at least $1 - 2/n^3$.

**Corollary 3.11.** Assume $k \geq 96 \ell / \epsilon^2$. Let $T_i$ be the set that won the Smooth-Compare tournament. Then, with probability at least $1 - 1/n^2$:

$$f(T) \geq (1 - \epsilon/3) \max_{j \in [1/\delta]} f(T_j).$$

**Proof.** Since Smooth-Compare is called to make $1/\delta$ comparisons, and we proved that $\forall i, j \in [1/\delta]$ the call Smooth-Compare($(\{T_i, T_j\}, H_{ij})$) returns $T_i$ as long as $f(T_i) \geq (1 - \epsilon \delta / 3) f(T_j)$. We get:

$$f(T_i) \geq (1 - \epsilon \delta / 3)^{1/\delta} \max_{j \in [1/\delta]} f(T_j) \geq (1 - \epsilon / 3) \max_{j \in [1/\delta]} f(T_j).$$

**Theorem 3.1.** Let $f : 2^N \to \mathbb{R}$ be a monotone submodular function. For any fixed $\epsilon > 0$, when $k \geq 1152 \log n / \epsilon^2$, then given access to a noisy oracle whose noise is an exponentially decaying tail distribution, the Slick-Greedy algorithm returns a set which is a $(1 - 1/e - \epsilon)$ approximation to $\max_{S: |S| \leq k} f(S)$, with probability at least $1 - 1/n$.

**Proof.** From Lemma 3.8 we know that when $k > 36 \ell / \epsilon^2$ there exists a smoothing set $H_i$ for which $\OPT_{H_i} \geq (1 - \epsilon / 3) \OPT$. From Lemma 3.6 we know that with probability at least $1 - 1/n^3$ Slick-Greedy returns a set $T_i = S_i \cup H_i$ which is a $1 - 1/e - \epsilon / 3$ approximation to $\OPT_{H_i}$. Therefore with probability $1 - 1/n^3$:

$$f(T_i) \geq (1 - 1/e - 2\epsilon / 3) \OPT$$

When $k \in \left(\frac{4608 \log \log n}{\epsilon^2}, \frac{1152 \log n}{\epsilon^2}\right)$ we use $\ell = 48 \log \log n$ and then $k \geq 96 \ell / \epsilon^2$. In the case that $k \geq 1152 \log n / \epsilon^2$ we have that $\ell = 12 \log n$ and in this case too $k \geq 96 \ell / \epsilon^2$. Therefore, the conditions for Corollary 3.11 hold, and we know that with probability $1 - 1/n^2$ the set $T_i$ returned by Slick-Greedy respects $f(T_i) \geq f(T_i)$. Taking a union bound on these events we get our result.

### 4 Optimization for Small $k$

When $k$ is small we cannot use the smoothing technique from the previous section, since it requires including the smoothing set of size $\Theta(\log \log n)$ in the solution. In this section we describe the **sampled mean method** which can be applied to $k \in \mathcal{O}(\log \log n)$. This method leads to an approximation arbitrarily close to $1 - 1/e$ for any $k \in \Omega(1/\epsilon)$. Somewhat counter-intuitively, when $k$ is constant the optimization problem becomes strictly harder. A variant gives a $1 - 1/e - \epsilon$ for $k > 3$ and $k/(k + 1) - \epsilon$. In the following section we show lower bounds for small values of $k$ and when $k = 1$ that no algorithm can obtain an expected approximation ratio better than $1/2 + o(1)$.

#### 4.1 Combinatorial averaging

Similar to the Slick-Greedy algorithm, the sampled-mean method is based on averaging sets to find elements whose marginal contribution is high, which can then be greedily added to the solution. Unlike the Slick-Greedy algorithm however, the idea here is to select smoothing sets that are intuitively close to the elements whose marginal contribution we estimate. Borrowing again from continuous optimization, the idea in this section is to define a small ball around a point. For this purpose, unlike the previous algorithm, we will add sets of constant size $c \in \mathcal{O}(1/\epsilon)$ to the solution each time, and not just single elements. Unlike the previous section, for a given set we will generate multiple smoothed values and average them. We call these values the **mean contribution** of a set and we will define this concept shortly.
Smoothing. Consider some arbitrary ordering on the elements s.t. \( N = \{a_1, a_2, \ldots, a_n\} \). For a given set \( A \subset N \), let \( A_{-i} \) denote \( A \setminus \{a_i\} \). Throughout the rest of this section we will work on sets of size \( c \), where \( c \) is a constant that depends on \( \epsilon \) and will be fixed later. For given sets \( S \) and \( A \), for every \( i \), the smoothing neighborhood of \( A_{-i} \) is \( \mathcal{H}(A_{-i}) = \{A_{-i} \cup \{a_j\} : a_j \in N \setminus (S \cup A)\} \). Thus:

\[
F(A_{-i}; S) = \frac{1}{n-c-|S|} \sum_{j \notin S \cup A} f(S \cup A_{-i} \cup a_j)
\]

\[
F_S(A_{-i}) = \frac{1}{n-c-|S|} \sum_{j \notin S \cup A} f_S(A_{-i} \cup a_j)
\]

And similarly, \( \bar{F}(A_{-i}; S) = \frac{1}{n-c-|S|} \sum_{j \notin S \cup A} \bar{f}(S \cup A_{-i} \cup a_j) \). For a given set \( A \) and every given \( a_i \in A \) we will apply smoothing arguments on each \( A_{-i} \) to show that the noisy smooth value is close to the true smooth value. We will compare between candidate solutions to be selected by the greedy algorithm using the mean smooth value which is simply an average of the smooth values of \( A_{-i} \), for all \( i \in [c] \). As we will soon see, for the set of size \( 1/\epsilon \) whose marginal contribution is highest, the mean value well approximates the true marginal contribution.

**Definition 4.1.** Suppose that we’ve already committed to a set \( S \). For a set \( A \) of fixed size \( c \), \( a_i \in A \) and \( a_j \in N \setminus (S \cup A) \) we define \( A_{ij}(S) := (A \setminus \{a_i\}) \cup \{a_j\} \). When \( S \) is clear from context, we will use the shorthand \( A_{ij} \). The mean smooth value, noisy mean smooth value and marginal smooth value of a set \( A \) to an existing set \( S \) are, respectively:

- \( \phi(A; S) := \frac{1}{c} \sum_{i \in A} F(A_{-i}; S) = \frac{1}{c(n-c-|S|)} \sum_{i \in A} \sum_{j \notin A \cup S} f(S \cup A_{ij}) \)
- \( \bar{\phi}(A; S) := \frac{1}{c} \sum_{i \in A} \bar{F}(A_{-i}; S) = \frac{1}{c(n-c-|S|)} \sum_{i \in A} \sum_{j \notin A \cup S} \bar{f}(S \cup A_{ij}) \)
- \( \phi_S(A) := \frac{1}{c} \sum_{i \in A} F_S(A_{-i}) = \frac{1}{c(n-c-|S|)} \sum_{i \in A} \sum_{j \notin A \cup S} f_S(A_{ij}) \)

### 4.2 The sampled mean algorithm

The SM-GREEDY begins with the empty set \( S \) and at every iteration considers subsets of size \( c \in O(1/\epsilon) \) to add to \( S \). At every iteration, the algorithm first takes the set \( A \) which maximizes the noisy mean smooth value. After taking \( A \) the algorithm then considers all possible subsets \( A_{ij} \) and takes the set whose noisy value is largest. We describe the algorithm formally below.

**Algorithm 4 SM-GREEDY**

**Input:** budget \( k \), precision \( \epsilon > 0 \), \( c \in O(1/\epsilon) \)

1. \( S \leftarrow \emptyset \)
2. \( \textbf{while } |S| < c \cdot \lfloor k/c \rfloor \textbf{ do} \)
3. \( A \leftarrow \arg\max_{|A| = c} \bar{\phi}(A; S) \)
4. \( S \leftarrow S \cup \arg\max_{i \in A, j \in N \setminus (S \cup A)} \bar{f}(S \cup A_{ij}) \)
5. \( \textbf{end while} \)
6. \( \textbf{return } S \)

At a high level, the major steps in the analysis can be described as follows.

1. We begin by giving guarantees on smoothing. In Lemma 4.2 we motivate the idea behind the mean contribution. We show that when \( c = 1/\epsilon \) the mean contribution is a \( 1-\epsilon \) approximation to \( f_S(A^*) \) where \( A^* \in \arg\max_{|A| = c} f_S(A) \). Using Claim 4.3, in Lemma 4.5 we then show that the set selected in each iteration is with high probability an arbitrarily good approximation to the set with maximal (non-noisy) smooth mean contribution.
2. Once done with smoothing arguments, in Lemma 4.6 we prove that if the marginal contribution \( f_S(\hat{A}) \) of the set \( \hat{A} \) we select at every iteration is close to the mean smooth marginal contribution \( \phi_S(A) \) we obtain an approximation arbitrarily close to \( 1 - 1/e \). This suffices for an approximation guarantee that holds in expectation.

3. The last step is essentially Lemma 4.13 where we show that taking \( \hat{A} \in \arg\max_{i,j} \tilde{f}(S \cup A_{ij}) \) in line 4 of the algorithm gives us, with high probability, that the marginal contribution of the set we select is close to its smooth marginal contribution. We can therefore invoke Lemma 4.6 and obtain the approximation guarantee.

### 4.3 Smoothing guarantees

**Lemma 4.2.** For a fixed set \( S \subset N \), let \( A^* \in \arg\max_{|A|=c} f_S(A) \). Then:

\[
\left( 1 - \frac{1}{c} \right) f_S(A^*) \leq \phi_S(A^*) \leq f_S(A^*)
\]

**Proof.** By the maximality of \( A^* \) we have that \( f(A^*) \geq f(A_{ij}) \) for any \( i,j \) since \( A_{ij}^* \) is generated by replacing \( a_i \in A^* \) with \( a_j \in N \setminus A^* \). Therefore, the average of all \( A_{ij}s \) is upper bounded by \( f_S(A^*) \).

For the lower bound, consider some arbitrary ordering on the elements \( a_1, \ldots, c \in A^* \). From the diminishing returns property of submodular functions we have that for any \( i \in [c] \):

\[
f_{S \cup A^*_{-i}}(a_i) = f(S \cup A^*_{-i} \cup a_i) - f(S \cup A^*_{-i}) \leq f(S \cup \{a_1, \ldots, a_i\}) - f(S \cup \{a_1, \ldots, a_{i-1}\})
\]

Thus:

\[
\sum_{i=1}^{c} f_{S \cup A^*_{-i}}(a_i) \leq \sum_{i=1}^{c} (f(S \cup \{a_1, \ldots, a_i\}) - f(S \cup \{a_1, \ldots, a_{i-1}\})) = f_S(A^*)
\]

Therefore, from monotonicity and submodularity we get:

\[
\phi_S(A^*) = \frac{1}{c(n - c - |S|)} \sum_{j=1}^{n-c-|S|} \sum_{i=1}^{c} f_S(A^*_{ij}) \\
\geq \frac{1}{c} \sum_{i=1}^{c} f_S(A^*_{-i}) \\
= \frac{1}{c} \sum_{i=1}^{c} \left( f_S(A^*_i \cup a_i) - f_{S \cup A^*_{-i}}(a_i) \right) \\
= \frac{1}{c} \sum_{i=1}^{c} f_S(A^*) - \frac{1}{c} \sum_{i=1}^{c} f_{S \cup A^*_{-i}}(a_i) \\
\geq f_S(A^*) - \frac{1}{c} f_S(A^*) \\
= \left( 1 - \frac{1}{c} \right) f_S(A^*). \\
\]

We would now like to employ the smoothing arguments from Section 2. We will show that \( \phi(A) \) where \( A \) is selected by the SM-GREEDY algorithm in line 3 is a good approximation to \( \phi_S(A^*) \), where \( A^* \in \arg\max_{|B|=c} f_S(B) \). From Lemma 4.2 above, this implies that \( \phi_S(A) \) well approximates \( f_S(A^*) \). To do so, we use the next claim which essentially relies on bounding the variation of the smoothing neighborhoods \( \mathcal{H}(A^*_{-i}) \), for almost all sets \( A^*_{-i} \).
Claim 4.3. Let \( A^* \in \arg \max_{B: |B| = c} f_S(B) \), \( c \geq 4/\epsilon \). Then:

\[
\frac{1}{c} \sum_{i=1}^{c} \max \left\{ 0, 1 - 2v_S(\mathcal{H}(A^*_{-i})) \cdot t^{-1/4} \right\} F_S(A^*_{-i}) \geq (1 - \epsilon) f_S(A^*)
\]

Proof. To bound the average variation of the sets \( \{A^*_{-i}\}_{i=1}^{c} \) we argue that at most one set \( A^*_{-i} \) will be s.t. \( f_S(A^*_{-i}) < f_S(A^*)/2 \). To see this, assume for purpose of contradiction there are \( A^*_{-i} \) and \( A^*_{-j} \) for which \( f_S(A^*_{-i}) \leq f_S(A^*_{-j}) < f_S(A^*)/2 \), then since \( A^* = A^*_{-i} \cup A^*_{-j} \) we get a contradiction:

\[
f_S(A^*) = f_S(A^*_{-i} \cup A^*_{-j}) \leq f_S(A^*_{-i}) + f_S(A^*_{-j}) < 2 \cdot \frac{f_S(A^*)}{2} = f_S(A^*)
\]

We therefore have at least \( c-1 \) sets such that each \( A^*_{-i} \) in this set respects \( f_S(A^*_{-i}) \geq f_S(A^*)/2 \). Call these sets bounded. For any such bounded set \( A^*_{-i} \), since \( A^*_{-i} \subseteq A^*_{ij} \) for any \( j \in \mathbb{N} \setminus (S \cup A^*) \), monotonicity implies:

\[
\min_{A^*_{ij} \in \mathcal{H}(A^*_{-i})} f_S(A^*_{ij}) \geq \frac{f_S(A^*)}{2}
\]

For a given set \( A^*_{-i} \) note that for every \( j \), every set \( A^*_{ij} \in \mathcal{H}(A^*_{-i}) \) respects \( f_S(A^*_{ij}) \leq f_S(A^*) \) due to the maximality of \( A^* \). Thus for any bounded set \( A^*_{-i} \):

\[
v_S(\mathcal{H}(A^*_{-i})) = \max_{A^*_{ij} \in \mathcal{H}(A^*_{-i})} \min_{A^*_{ij} \in \mathcal{H}(A^*_{-i})} \frac{f_S(A^*_{ij})}{f_S(A^*)} \leq \frac{f_S(A^*)}{f_S(A^*)/2} = 2
\]

Let \( l \) be the index of the set \( A^*_{-i} \) with the lowest value \( f_S(A^*_{-i}) \). Our discussion above implies that this is the only set whose variation may not be bounded from above by 2. Assume \( n \) sufficiently large s.t. \( t \geq 2^{12}/\epsilon^4 \). We therefore get:

\[
\frac{1}{c} \sum_{i=1}^{c} \left( \max\{0, 1 - 2v_S(\mathcal{H}(A^*_{-i})) t^{-\frac{1}{4}} \} \right) F_S(A^*_{-i}) \geq \frac{1}{c} \sum_{i \neq l} \left( \max\{0, 1 - 2v_S(\mathcal{H}(A^*_{-i})) t^{-\frac{1}{4}} \} \right) F_S(A^*_{-i}) \quad (13)
\]

\[
\geq \frac{1}{c} \sum_{i \neq l} \left( 1 - 4t^{-\frac{1}{4}} \right) F_S(A^*_{-i}) \quad (14)
\]

\[
\geq \frac{1}{c} \sum_{i \neq l} \left( 1 - 4t^{-\frac{1}{4}} \right) f_S(A^*_{-i}) \quad (15)
\]

\[
\geq \left( 1 - 4t^{-\frac{1}{4}} \right) \frac{1}{c} \left( \sum_{i=1}^{c} f_S(A^*_{-i}) - f_S(A^*_{-l}) \right) \quad (16)
\]

\[
\geq \left( 1 - 4t^{-\frac{1}{4}} \right) \frac{1}{c} \left( (c-1) f_S(A^*) - f_S(A^*_{-l}) \right) \quad (17)
\]

\[
\geq \left( 1 - 4t^{-\frac{1}{4}} \right) \frac{1}{c} \left( (c-1) f_S(A^*) - f_S(A^*) \right) \quad (18)
\]

\[
\geq \left( 1 - 4t^{-\frac{1}{4}} \right) \left( \frac{c-2}{c} \right) f_S(A^*) \quad (19)
\]

\[
\geq \left( \frac{c-2}{c} - 4t^{-\frac{1}{4}} \right) f_S(A^*) \quad (20)
\]

\[
\geq (1 - \epsilon) f_S(A^*) \quad (21)
\]

The inequality (14) is justified by the bound we established on bounded sets; (15) is due to monotonicity of \( f_S \), since \( F_S(A^*_{-i}) \) is an average of the marginal contribution over all possible \( A^*_{ij} \), which
is a superset of $A^*_i$; (17) is due to an argument in the proof of Lemma 4.2; (18) is due to the optimality of $A^*$; (21) is due to the assumption on the parameters in the statement of the claim.

Similarly to the definition of $\epsilon$-relevant iterations from the previous section, we introduce the definition of $\epsilon$-significant iterations to employ smoothing arguments in the next lemma.

**Definition 4.4.** Let $O \in \arg\max_{T:|T|\leq t} f(T)$. An iteration of SM-Greedy is $\epsilon$-significant if for the given set $S$ selected before the iteration we have that $f_S(O) \geq \epsilon f(O)$.

**Lemma 4.5.** Let $A \in \arg\max_{B:|B|=c} \tilde{\phi}(B; S)$ where $c \geq 16/\epsilon$, and assume that the iteration is $\epsilon/4$-significant. Then, with probability at least $1 - e^{-\Omega(t^{1/10})}$ we have that:

$$\phi_S(A) \geq (1 - \epsilon) \max_{B:|B|=c} \phi_S(B)$$

**Proof.** Let $A^* = \arg\max_{A:|A|=c} f_S(A)$ and let $B : |B| = c$ be such that $\phi_S(B) < (1 - \epsilon)\phi_S(A^*)$. Similar the proof from the previous section we will apply the smoothing arguments and show that with high probability $\tilde{\phi}(A^*; S) > \tilde{\phi}(B; S)$. By taking a union bound over all possible $O(nc^2)$ sets $B$ we will then conclude that the set whose smooth noisy contribution is largest must have smooth contribution at least factor of $(1 - \epsilon)$ of $A^*$, with high probability.

We will denote $\epsilon_1 = \epsilon$ and $\epsilon_2 = \epsilon/4$. Notice that the conditions of Claim 4.3 are met with $\epsilon_2$ and that the iteration is $\epsilon_2$-significant, which from submodularity implies $f_S(A^*) \geq \epsilon_2 \cdot f(S)/k$.

For a set $B_i \in B$, using Lemma 2.5, for $t = n - c - |S|$, when $\omega$ denotes the highest realized value of a noise multiplier, we know that for $\lambda \in [0, 1)$ with probability $1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)}$:  

$$\tilde{\phi}(B; S) = \frac{1}{c} \sum_{i=1}^c \tilde{F}(B_{-i}; S) 
\leq \frac{1}{c} \sum_{i=1}^c (1 + \lambda) \mu \cdot \left( f(S) + F_S(B_{-i}) + 3t^{-1/4} \max_{B_{ij} \in \mathcal{H}(B_{-i})} f_S(B_{ij}) \right) 
\leq (1 + \lambda) \mu \cdot \left( f(S) + 3t^{-1/4} \max_{B_{ij} \in \mathcal{H}(B_{-i})} f_S(B_{ij}) + \frac{1}{c} \sum_{i=1}^c F_S(B_{-i}) \right) 
\leq (1 + \lambda) \mu \cdot \left( f(S) + 3t^{-1/4} f_S(A^*) + \frac{1}{c} \sum_{i=1}^c F_S(B_{-i}) \right) 
\leq (1 + \lambda) \mu \cdot \left( f(S) + 3t^{-1/4} f_S(A^*) + \phi(B; S) \right) 
\leq (1 + \lambda) \mu \cdot \left( f(S) + 3t^{-1/4} f_S(A^*) + (1 - \epsilon_1) \phi(A^*; S) \right) 
\leq (1 + \lambda) \mu \cdot \left( f(S) + 3t^{-1/4} f_S(A^*) + (1 - \epsilon_1) f_S(A^*) \right) 
= (1 + \lambda) \mu \cdot \left( f(S) + f_S(A^*) \left( 3t^{-1/4} + (1 - \epsilon_1) \right) \right)$$

We now need to argue that $\phi(A^*; S)$ is sufficiently large to beat $\phi(B; S)$. Assuming $n$ is sufficiently large s.t. $t \geq 2^{20}/\epsilon^4$, from lemmas 2.4 and 4.3 we know that for $\lambda \in [0, 1)$ w.p. $1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)}$:

$$\tilde{\phi}(A^*; S) = \frac{1}{c} \sum_{i=1}^c \tilde{F}(A^*; S) 
> (1 - \lambda) \mu \cdot \left( f(S) + \frac{1}{c} \sum_{i=1}^c \left( 1 - 2v(\mathcal{H}(A^*_i)) \cdot t^{-1/4} \right) \cdot F_S(A^*) \right) 
> (1 - \lambda) \mu \cdot \left( f(S) + (1 - \epsilon_2) f_S(A^*) \right)$$

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Lemma 4.6. Condition suffices for an approximation guarantee that holds in expectation. We will soon after set of size 10

\[ \phi(A^*; S) - \phi(B; S) \geq \mu \left( (1 - \lambda) \cdot (f(S) + (1 - \epsilon_2) f_S(A^*)) - (1 + \lambda) \cdot \left( f(S) + f_S(A^*) \left( 3t^{-1/4} + (1 - \epsilon_1) \right) \right) \right) \]

\[ \geq \mu \left( (1 - \lambda)(1 - \epsilon_2) f_S(A^*) - 2\lambda f(S) - (1 + \lambda) \left( 3t^{-1/4} + (1 - \epsilon_1) \right) f_S(A^*) \right) \]

\[ \geq \mu \left( (1 - \lambda)(1 - \epsilon_2) f_S(A^*) - 2\lambda f(S) - (1 + \lambda) \left( 3t^{-1/4} + (1 - \epsilon_1) \right) f_S(A^*) \right) \]

\[ \geq \mu \cdot f_S(A^*) \left( 1 - \lambda - \epsilon_2 - \frac{2\lambda k}{\epsilon_2} - \epsilon_2 - \lambda \epsilon_2 - 1 - \lambda + \epsilon_1 \right) \]

\[ > \mu \cdot f_S(A^*) \left( \epsilon_1 - 3\epsilon_2 - \lambda \left( \frac{2k}{\epsilon_2} \right) \right) \]

For any \( \lambda \leq \epsilon^2/2k \) the difference above is strictly positive. Conditioning on \( \omega \) being bounded from above by \( t^{1/5} \) which happens with probability \( 1 - e^{-\Omega(t^{1/5}/\log t)} \), since \( k \in O(\log \log n) \) we that the result holds with probability at least \( 1 - e^{-\Omega(t^{1/10})} \).

\[ \square \]

4.4 Approximation guarantee

We begin by giving an approximation guarantee, assuming that the marginal contribution of the set of size \( c \) we select at every iteration is close to its mean smooth marginal contribution. This condition suffices for an approximation guarantee that holds in expectation. We will soon after discuss how to obtain a guarantee that holds with high probability.

**Lemma 4.6.** Assume that at every \( \delta/4 \)-significant iteration of SM-GREEDY when the set selected at previous iterations is \( S \) and the selected is \( A \) we have that: \( f_S(A) \geq (1 - \delta) \max_{A: |A| = c} \phi_S(A) \), for \( \delta > 0 \) with probability \( p \). Assume that \( k > c/\delta \), \( c \geq 16/\delta \). Let \( S \) be the set of elements selected in all the iterations of the algorithm SM-GREEDY. Then, with probability \( \geq (1 - k(1 - p))(1 - 1/n^2) \):

\[ f(S) = (1 - 1/e - 5\delta)OPT \]

**Proof.** We condition on the high probability event in Lemma 4.5 and that every one of the \( k \) iterations produces \( f_S(A) \geq (1 - \delta) \max_{A: |A| = c} \phi_S(A) \). Since there are less than \( k \) iterations in which a set of elements \( A \) is added to the solution, by a union bound the result will hold with probability at least \( (1 - k(1 - p))(1 - 1/n^2) \). We assume that \( n \) is sufficiently large s.t. \( t \geq 2^{20}/\delta^4 \).

First, we rely on the smoothing argument which assumes that the iterations are \( \delta \)-significant. Therefore, we will compare against \( (1 - \delta) \) of the optimal value: let \( k \) be the last \( \delta \)-significant iteration and \( O \) be the subset of size \( k \) of the optimal solution whose value is largest. By submodularity:

\[ f(O) \geq (1 - \delta)OPT \]

Second, we argue that optimizing over sets of size \( c \) rather than singletons is inconsequential when \( k > c/\epsilon \). To be convinced, notice that when the algorithm selects \( c \) elements in every iteration the total number of elements selected will be \( k' > k - c \). Let \( O' \in \arg\max_{T: |T| \leq k'} f(T) \). As in previous arguments, from submodularity we have that: \((1 - c/k)f(O) \leq f(O')\). Since \( k > c/\epsilon \) we have that:

\[ f(O') > (1 - \delta)f(O) > (1 - 2\delta)OPT \]
We will henceforth analyze the algorithm against $O'$. In a similar manner to the analysis of the greedy algorithm which selects singletons at every stage $i \in [k]$, we can analyze the greedy algorithm which selects sets of size $c$ at every stage $i \in [k'/c]$. To ease notation assume $[k'/c] = k'/c$.

For a given stage of the algorithm, assume the set $S$ has been previously selected and that a set $A$ is being added into the solution. Let $O^* = \arg \max_{T \subseteq O', |O| = c} f_S(O')$ and $A^*= \arg \max_{B:|B|=c} f_S(B)$.

$$f_S(A) \geq (1-\delta) \phi_S(A)$$  \hfill (24)
$$> (1-2\delta) \phi_S(A^*)$$  \hfill (25)
$$> (1-3\delta) f_S(A^*)$$  \hfill (26)
$$> (1-3\delta) f_S(O^*)$$  \hfill (27)
$$> (1-3\delta) \frac{c}{k'} f_S(O')$$  \hfill (28)
$$= (1-3\delta) \frac{c}{k'} (f(O' \cup S) - f(S))$$  \hfill (29)
$$\geq (1-3\delta) \frac{c}{k'} (f(O') - f(S))$$  \hfill (30)

The inequality (24) is due to the assumption in the statement of the Lemma; (25) is due to Lemma 4.5 applied with $\epsilon = \delta$; (26) is due to Lemma 4.2 and $c \geq 1/\delta$; (27) is by the maximality of $A^*$; (28) is due to subadditivity.

An inductive argument similar to that in proof of Lemma 3.6 from the previous section can show $f(\tilde{S}) \geq (1-1/e-3\delta) f(O')$. Since we lose $2\delta$ from (23) this concludes our proof. \hfill \blacksquare

### 4.4.1 From expectation to high probability

Lemma 4.5 gives us a bound on the performance of the mean marginal contribution (measured by $\phi$), but not the actual marginal contribution, which is determined by the set we actually choose to add to the solution. Note that adding the set which maximizes the mean smooth value can easily lead to an arbitrarily bad approximation. Choosing a random $A_{ij}$ would have given us the correct expected marginal contribution, but then the result would not be with high probability of success. It remains to show that after we selected the set $A$ which maximizes $\phi_S(A)$, choosing the set which maximizes $\tilde{f}(S \cup A_{ij})$ is a good approximation of $\phi_S(A)$, with high probability.

**High-level overview to show high probability guarantee.** Let $A^*$ be the set of the largest marginal contribution and $A$ be the set selected by the algorithm at an iteration. That is, $A^* \in \arg \max_{B:|B|=c} f_S(B)$ and $A \in \arg \max_{B:|B|=c} \tilde{f}(B)$. We will define two kinds of sets in $\{A_{ij}\}_{i \in [c], j \in N \setminus (S \cup A)}$, called **good** and **bad**. The good sets are those whose true marginal contribution is at most $1-2\epsilon$ from $f_S(A^*)$ and the bad sets are those whose marginal contribution is at least $1-3\epsilon$ from $f_S(A^*)$. Our goal would be to prove that with high probability no bad set can be returned by the algorithm. To do so, we will prove that with high probability a good set $A'$ beats a bad set $A''$ when comparing $\tilde{f}(S \cup A')$ with $\tilde{f}(S \cup A'')$. This will be done using the following steps:

1. After defining good and bad sets, in Claim 4.8 we show that for $A \in \arg \max_{B:|B|=c} \phi(B)$, at least half of the sets in $\{A_{ij}\}_{i \in A, j \in N \setminus (S \cup A)}$ are good, and at most half are bad;

2. Next, we define two thresholds: $M_g$ and $M_b$. Intuitively, $M_g$ is the value of the noise multiplier that one of the good sets should obtain, and $M_b$ is an upper on the value of the noise multiplier that any one of the bad sets can obtain;

---

As an example, consider an instance with $n-1$ complementary elements $M$ for whom for any $S \subseteq M$ the function evaluates to $f(S) = \alpha$ for some arbitrarily high value $\alpha$, and an additional subset of elements $A$ s.t. $f(A) = \epsilon$ and for any $S \subseteq M$ we have $f(S \cup A) = M + \epsilon$, for some arbitrarily small $\epsilon > 0$. The sampled mean of $A$ is maximal, while its value is arbitrarily small.
3. We will then show in Lemma 4.12 that $M_0 \geq (1 - \gamma)M_b$, for any $\gamma = \Theta(1/\log \log n)$ to be fixed later. Notice that this then implies that the noisy value of a good set is then larger than the noisy value of a bad set. This lemma is quite technical, and it is where we fully leverage the property of generalized exponential tail distribution and the fact that $k \in O(\log \log n)$.

4. Putting (31) and (32) together we get that for sufficiently large \( \epsilon > 0 \)

**Claim 4.8.** For a given set $S$, let $A^* \in \arg\max_{B:|B|=c} f_S(B)$, $A \in \arg\max_{B:|B|=c} \phi(B; S)$, and $A = \{A_{ij}\}_{i,j \in A}$. For a fixed $\epsilon > 0$:

- $A_{ij} \in A$ is $\epsilon$-good if $f_S(A_{ij}) \geq (1 - 2\epsilon)f_S(A^*)$; let $\epsilon$-good$(A)$ denote all $\epsilon$-good $A_{ij} \in A$;
- $A_{ij} \in A$ is $\epsilon$-bad if $f_S(A_{ij}) \leq (1 - 3\epsilon)f_S(A^*)$; let $\epsilon$-bad$(A)$ denote all $\epsilon$-bad $A_{ij} \in A$.

**Definition 4.7.** For a given set $S$, let $A^* \in \arg\max_{B:|B|=c} f_S(B)$, $A \in \arg\max_{B:|B|=c} \phi(B; S)$, and $A = \{A_{ij}\}_{i,j \in A}$. For a fixed $\epsilon > 0$:

- $|\epsilon$-good$(A)| \geq \frac{c(n-c-|S|)}{2}$;
- $|\epsilon$-bad$(A)| \leq \frac{c(n-c-|S|)}{2}$.

**Proof.** Since the sets $A_{ij}$ are distinct both $\epsilon$-good$(A)$ and $\epsilon$-bad$(A)$ contain no repetitions and we can argue about their size. To lower bound the size of $\epsilon$-good$(A)$, let $A^* \in \arg\max_{A:|A|=c} f_S(A)$. When the iteration is $\epsilon/8$-significant, from Lemma 4.5 we know that with exponentially high probability:

\[ \phi_S(A) \geq (1 - \epsilon/2) \phi_S(A^*) \]

When $c \geq 2/\epsilon$, from Lemma, we know that:

\[ \phi_S(A^*) \geq (1 - \epsilon/2) f_S(A^*) \]

Denoting $m = c(n - c - |S|)$, we get with exponentially high probability:

\[ \phi_S(A) = \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{c} f_S(A_{ij}) \geq (1 - \epsilon) f_S(A^*) \]  \hspace{1cm} (31)

In addition, due to the maximality of $A^*$ we have that $f_S(A_{ij}) \leq f_S(A^*)$ for every $i, j$. Therefore:

\[ \sum_{j=1}^{m} \sum_{i=1}^{c} f_S(A_{ij}) \leq |\epsilon$-good$(A)| \cdot f_S(A^*) + (m - |\epsilon$-good$(A)|) \cdot (1 - 2\epsilon) f_S(A^*) \]  \hspace{1cm} (32)

Putting (31) and (32) together we get that for sufficiently large $n$, with probability at least $1 - 1/n^{10}$:

\[ m(1 - \epsilon) f_S(A^*) \leq (|\epsilon$-good$(A)| + (m - |\epsilon$-good$(A)|)(1 - 2\epsilon)) f_S(A^*) \]

Rearranging and using $m = c(n - c - |S|)$ we get that $|good| \geq c(n - c - |S|)/2$. Since there are a total of $c(n - c - |S|)$ it follows that $|\epsilon$-bad$(A)| \leq c(n - c - |S|)$ as required.

**Definition 4.9.** Let $\rho_D(x)$ denote the probability density function of $D$. For a set $S : |S| = O(\log n)$, $c > 0$, $\gamma > 0$, we define $M_g$ and $M_b$ as:

- $\int_{M_b}^{\infty} \rho_D(x) dx = \frac{2}{c(n-c-|S|)\log n}.$
\[ \int_{M_b}^{\infty} p_D(x) \, dx = \frac{2 \log n}{c(n - c - |S|)}. \]

The following claim immediately follows from the definition, yet it is still useful to specify explicitly. The claim considers \( c(n - c - |S|)/2 \) samples since this is an upper and lower bound on \(|\varepsilon\text{-good}(A)|\) and \(|\varepsilon\text{-bad}(A)|\). Therefore the claim gives us the likelihood that the largest noise multiplier of \( \varepsilon\text{-bad}(A) \) does not exceed \( M_b \) and that at least one set from \( \varepsilon\text{-good}(A) \) exceeds \( M_g \).

**Claim 4.10.** For a fixed set \( S \) and \( A \in \text{arg max}_{B:|B|=c} \tilde{\phi}(B; S) \), let \( m = c(n - c - |S|) \) and consider \( m/2 \) independent samples from the noise distribution. Then:

\begin{itemize}
  \item \( \Pr[\max\{\xi_1, \ldots, \xi_{m/2}\} \leq M_b] > (1 - \frac{2}{\log n})^{m/2} \)
  \item \( \Pr[\max\{\xi_1, \ldots, \xi_{m/2}\} \geq M_g] > 1 - 2/n. \)
\end{itemize}

**Proof.** For a single sample \( \xi \) from \( \mathcal{D} \), we have that:

\[ \Pr[\xi \leq M_b] = 1 - \frac{2}{m \log n} \]

If we take \( m/2 \) independent samples \( \xi_1, \ldots, \xi_{m/2} \), the probability they are all bounded by \( M_b \) is:

\[ \Pr[\max\{\xi_1, \ldots, \xi_{m/2}\} \leq M_b] \geq \left( 1 - \frac{2}{m \log n} \right)^{m/2} > \left( 1 - \frac{2}{\log n} \right)^{m/2} \]

In the case of \( M_g \), the probability that a single sample \( \xi \) taken from \( \mathcal{D} \) is at most \( M_g \) is equal to:

\[ \Pr[\xi \leq M_g] = 1 - \frac{2 \log n}{m} \]

If we take independent samples \( \xi_1, \ldots, \xi_{m/2} \), the probability they are all bounded by \( M_g \) is:

\[ \Pr[\max\{\xi_1, \ldots, \xi_{m/2}\} \leq M_g] = \left( 1 - \frac{2 \log n}{m} \right)^{m/2} < \frac{2^{\log n}}{2^{\log n}} = \frac{2}{n} \]

And accordingly the probability that at least one of these samples is greater than \( M_g \) is:

\[ \Pr[\max\{\xi_1, \ldots, \xi_{m/2}\} \geq M_g] > 1 - 2/n. \]

The Lemma 4.12 below relates \( M_g \) and \( M_b \) assuming that \( \mathcal{D} \) is has a generalized exponential tail. This lemma makes the result applicable for Exponential and Gaussian distributions, and it fully leverages the fact that \( k \in O(\log \log n) \). The lemma is quite technical, and we first prove the much simpler case where the distribution is bounded, which applies to the case of uniform distributions.

**Lemma 4.11.** Assume \( \mathcal{D} \) has a generalized exponential tail and that \( \mathcal{D} \) is bounded. Let \( x_0 \) be the infimum s.t. \( \mathcal{D} = 1 \) and assume \( \rho_D(x_0) \neq 0 \). Then: \( (1 - \gamma)M_b \leq M_g, \forall \gamma \in \Omega(1/\log \log n) \).

**Proof.** Since \( x_0 \) is the infimum number such that \( \mathcal{D} = 1 \), we have that \( M_b \leq x_0 \). Let \( x_g = (1 - \gamma)x_0 \). Since we know that \( x_0 \) is an infimum and \( \rho_D(x_0) \neq 0 \) it must be that \( \mathcal{D}(x_g) = 1 - \delta \) for some \( \delta = \Omega(1/\log \log n) \). But in this case, if we take \( n > \log n/\gamma \delta \) we get that \( M_g > x_g \), and hence \( M_g \geq (1 - \gamma)M_b \).

**Lemma 4.12.** If \( \mathcal{D} \) has a generalized exponential tail then \( (1 - \gamma)M_b \leq M_g, \forall \gamma \in \Omega(1/\log \log n) \).

**Proof.** The proof follows three stages:

1. We use properties of \( \mathcal{D} \) to argue upper and lower bounds for \( \rho_D(x) \);
2. We show an upper bound \( M \) on \( M_b \);

3. We show that integrating a lower bound of \( \rho_D(X) \) from \( (1 - \gamma)M \) to \( \infty \), yields a probability mass at least \( \frac{\log n}{\gamma (n - c - |S|)} \). Now suppose for contradiction that \( M_g < (1 - \gamma)M_b \), we would get that \( \int_{M_g}^{\infty} \rho_D(x) \) is strictly greater than \( \frac{\log n}{\gamma (n - c - |S|)} \), which contradicts the definition of \( M_g \).

We now elaborate each on stage. Recall that by definition of \( D \) for \( x \geq x_0 \), we have that

\[
\rho_D(x) = e^{-g(x)}, \quad \text{where } g(x) = \sum_i a_i \alpha_i \quad \text{and that we do not assume that all the } \alpha_i \text{’s are integers, but only that } \alpha_0 \geq \alpha_1 \geq \ldots \text{, and that } \alpha_0 \geq 1. \quad \text{We do not assume anything on the other } \alpha_i \text{ values.}
\]

For the first stage we will show that for every \( g(x) \), there exists \( n_0 \) such that for any \( n > n_0 \) and \( x \geq \left( \frac{\log n}{2a_0} \right)^{1/\alpha_0} \) we have that for \( \beta = \gamma / 100 < 1/100 \):

\[
(1 + \beta)a_0x^{\alpha_0 - 1}e^{-\alpha_0 - 1} \leq e^{-\alpha_0 - 1} \leq \rho_D(x) \leq (1 - \beta)a_0x^{\alpha_0 - 1}e^{-\alpha_0 - 1}
\]

We explain both directions of the inequality. To see \( a_0x^{\alpha_0 - 1}(1 + \beta)e^{-\alpha_0 - 1} \leq \rho_D(x) \) we first show:

\[
e^{-\alpha_0 - 1} \leq \rho_D(x)
\]

This holds since for sufficiently large \( n \), we have that:

\[
x \geq \frac{(\log n)^{1/\alpha_0}}{2a_0} \geq \left( \frac{2 \sum_i |a_i|}{\beta a_0} \right)^{\alpha_0 - \alpha_1}
\]

So the term \( \frac{\beta}{2}x^{\alpha_0} \) dominates the rest of the terms. We now show that:

\[
e^{-\alpha_0 - 1} \geq a_0x^{\alpha_0 - 1}(1 + \beta)e^{-\alpha_0 - 1}
\]

This is equivalent to:

\[
e^{\beta a_0/2x^{\alpha_0}} \geq a_0x^{\alpha_0 - 1}(1 + \beta)
\]

Which hold for \( x = \log \log^3 n \) and large enough \( n \).

The other side of the inequality is proved in a similar way. We want to show that:

\[
\rho_D(x) \leq (1 - \beta)a_0x^{\alpha_0 - 1}e^{-\alpha_0 - 1}
\]

Clearly for \( x > \log \log^3 n \) we have that \((1 - \beta)a_0x^{\alpha_0 - 1} > 1 \). Hence we just need to show that:

\[
\rho_D(x) \leq e^{-\alpha_0 - 1}
\]

But this holds for sufficiently large \( n \) s.t.:

\[
x \geq \frac{(\log n)^{1/\alpha_0}}{2a_0} \geq \left( \frac{\sum_i |a_i|}{\beta a_0} \right)^{\alpha_0 - \alpha_1}
\]

We now proceed to the second stage, and compute an upper bound on \( M_b \). Note that if

\[
\int_{M_b}^{\infty} \rho_D(x) = \int_M^\infty g(x)
\]

and for every \( x \geq M \) we have \( \rho_D(x) \leq g(x) \) then it must be that \( M \geq M_b \). Applying this to our setting, we bound \( \rho_D(x) \leq (1 - \beta)a_0x^{\alpha_0 - 1}e^{-\alpha_0 - 1} \) to get:

\[
\frac{1}{c(n - c - |S|) \log n} = \int_M^\infty (1 - \beta)a_0x^{\alpha_0 - 1}e^{-\alpha_0 - 1} \leq \frac{1}{M} e^{-\beta a_0x^{\alpha_0 - 1}} \leq \frac{1}{M} e^{-\alpha_0 - 1}
\]

= \frac{1}{M} e^{-\alpha_0 - 1} M^\alpha
\]
Taking the logarithm of both sides, we get:

\[-(1 - \beta) a_0 M^{a_0} = \log \frac{1}{c(n - c - |S|) \log n} \]
\[= -\log(c(n - c - |S|) \log n)\]

Multiplying by \(-1\), dividing by \((1 - \beta) a_0\) and taking the \(1/a_0\) root we get:

\[M = \left( \frac{\log(c(n - c - |S|) \log n)}{(1 - \beta) a_0} \right)^{a_0} \]

Note that \((1 - \gamma) M > \left( \frac{\log n}{2a_0} \right)^{1/a_0}\) and hence our bounds on \(\rho_D(x)\) hold for this regime.

We move to the third stage, and bound \(\int_{(1-\gamma)M}^{\infty} \rho_D(x)\) from below. If we show that: \(\int_{(1-\gamma)M}^{\infty} \rho_D(x)\) is greater than \(\frac{\log n}{\gamma c(n-c-|S|)}\), this implies that \(M_g \geq (1 - \gamma) M\), as \(M_g\) is defined as the value such that when we integrate \(\rho_D(x)\) from \(M_g\) to \(\infty\) we get exactly \(\frac{\log n}{\gamma c(n-c-|S|)}\). We show:

\[\int_{(1-\gamma)M}^{\infty} \rho_D(x) \geq (1 + \beta) a_0 a_0 x^{a_0} - e^{-(1 - \beta) a_0 x^{a_0}}\]
\[= e^{-(1 - \beta) a_0 x^{a_0}((1 - \gamma)M)}\]
\[= e^{-(1 - \beta) a_0 M^{a_0}(1 - \gamma)^{a_0}}\]
\[\geq e^{-(1 - \beta) a_0 M^{a_0}(1 - \gamma)}\]

However \(a_0 M^{a_0} = \left( \frac{\log(c(n - c - |S|) \log n)}{(1 - \beta)} \right)\). Since \(\beta < 0.1\) we have that \(\frac{1 + \beta}{1 - \beta} < 1 + 3\beta\). Substituting both expressions we get:

\[e^{-(1 + \beta) a_0 M^{a_0}(1 - \gamma)} \geq e^{(1 + 3\beta)(1 - \gamma) \log(c(n - c - |S|) \log n)}\]
\[= \left( \frac{1}{c(n - c - |S|) \log n} \right)^{(1 - \gamma)(1 + 3\beta)}\]
\[\geq \left( \frac{1}{c(n - c - |S|) \log n} \right)^{(1 - \gamma)/2}\]

Where we used that \(\beta = \gamma / 100\) and hence \((1 - \gamma)(1 + 3\beta) < 1 - \gamma / 2\). We now need to compare this to \(\frac{\sqrt{\log n}}{\gamma c(n-c-|S|)}\). To do this, note that:

\[\left( \frac{1}{c(n - c - |S|) \log n} \right)^{(1 - \gamma)/2} \geq \frac{1}{c(n - c - |S|) \log n} \]
\[\geq \frac{\sqrt{\log n}}{c(n - c - |S|) \log n} \]
\[\geq \frac{\gamma c(n - c - |S|)}{\log n} \]

Where \(n\) is large enough that \(\frac{\gamma}{2} \log(n - c - |S|) > \sqrt{\log n}\). This completes the proof, since \(M_g \geq (1 - \gamma) M \geq (1 - \gamma) M_0\) as required.

\[\square\]

**Lemma 4.13.** Consider an \(\epsilon/8\)-significant iteration of SM-GREEDY with a set \(S: S \in O(\log \log n)\), and let \(A = \arg \max_{i \in A, j \in N \setminus (S \cup A)} \hat{f}(S \cup A_{ij}), \) where \(A = \arg \max \tilde{f}(A; S)\) and \(c \geq 16/\epsilon\). For every \(\gamma = \Omega(1/\log \log n)\), with probability at least \(1 - 3/\log n\) we have that: \(f_S(A) \geq (1 - 3\epsilon) \phi_S(A)\).
Proof. We will use the above claims to argue that with probability at least \(1 - 4/\log n\) the noisy mean value of any set in \(\epsilon\cdot\text{bad}(A)\) is smaller than the largest noisy mean value of a set in \(\epsilon\cdot\text{good}(A)\). Since a bad set is defined as a set \(B\) for which \(f(B) \leq (1 - 3\epsilon)f_S(A^*)\) this implies that the set returned by the algorithm has value at least \((1 - 3\epsilon)f_S(A^*)\). Since for any set \(A: |A| = c\) we have that \(f_S(A^*)\) is an upper bound on \(\phi_S(A)\) will complete the proof.

We will condition on the event that \(|\epsilon\cdot\text{good}(A)| \geq \epsilon(n - c - |S|)/2\) which happens with probability at least \(1 - 1/n^{10}\) from Claim 4.8. Under this assumption, from Claim 4.10 we know that with probability at least \(1 - 2/n\) at least one of the noise multipliers of sets in \(\epsilon\cdot\text{good}(A)\) has value at least \(M_g\), and from Claim we know that \(M_g \geq (1 - \gamma)M_b\) for any \(\gamma \in \Theta(1/\log \log n)\). Thus:

\[
\max_{A_{ij} \in \epsilon\cdot\text{good}(A)} \tilde{f}(S \cup A_{ij}) = \max_{A_{ij} \in \epsilon\cdot\text{good}(A)} \xi_{A_{ij}} (f(S) + f_S(A_{ij})) \\
\geq M_g \cdot (f(S) + (1 - 2\epsilon)f_S(A^*)) \\
\geq (1 - \gamma)M_b \cdot (f(S) + (1 - 2\epsilon)f_S(A^*))
\]

Let \(B \in \arg\max_{C \in \epsilon\cdot\text{bad}(A)} \tilde{f}(S \cup C)\). From Claim 4.10 we know that with probability at least \(1 - 2/\log n\) all noise multipliers of sets in \(\epsilon\cdot\text{bad}(A)\) are at most \(M_b\). Thus:

\[
\tilde{f}(S \cup B) = \max_{A_{ij} \in \epsilon\cdot\text{bad}(A)} \tilde{f}(S \cup A_{ij}) = \max_{A_{ij} \in \epsilon\cdot\text{bad}(A)} \xi_{A_{ij}} f(S \cup A_{ij}) \leq M_b \cdot (f(S) + (1 - 3\epsilon)f_S(A^*))
\]

Let \(d\) be some constant such that \(|S| \leq d \log \log n\). Note that the iteration is \(\epsilon\)-significant, and therefore due to the maximality of \(A^*\) and since \(f(S) \leq \text{OPT}\) and the optimal solution has at most \(d \cdot \log \log n\) elements we have that:

\[
f_S(A^*) \geq \frac{\epsilon}{d \log \log n} f(S).
\]

Putting it all together and conditioning on all events we have with probability at least \(1 - 3/\log n\):

\[
\tilde{f}(S \cup \hat{A}) - \tilde{f}(S \cup B) \geq (1 - \gamma)M_b \cdot (f(S) + (1 - 2\epsilon)f_S(A^*)) - M_b \cdot (f(S) + (1 - 3\epsilon)f_S(A^*)) \\
\geq M_b (\epsilon f_S(A^*) - \gamma ((1 - 2\epsilon)f_S(A^*) + f(S))) \\
> M_b \cdot f_S(A^*) \left(\epsilon - \gamma \left(1 + \frac{d \cdot \log \log n}{\epsilon}\right)\right)
\]

Since Lemma 4.12 applies to any \(\gamma \in \Theta(1/\log \log n)\), we know that for any constant \(d\) there is a large enough value of \(n\) such that \(\gamma(1 + d \log \log n)/\epsilon < \epsilon\). Therefore the difference is strictly positive, implying that a bad set will not be selected by the algorithm which concludes our proof. \(\square\)

Theorem 4.14. For any monotone submodular function and any \(\epsilon > 0\), with probability \(1 - 4/\log n\) there is a \((1 - 1/\epsilon - \epsilon)\) approximation for \(\max_{S:|S| \leq k} f(S)\) for super-constant \(k = O(\log \log n)\), given access to a noisy oracle whose distribution has a generalized exponential tail.

Proof. Let \(\delta = \epsilon/5\) and set \(c \geq 16/\delta\). At any given \(\delta/8\)-significant iteration of SM-GREEDY from Lemma 4.13 we know that with probability at least \(1 - 3/\log n\) we have that \(f(\hat{A}) \geq 1 - \delta \phi_S(A)\), where \(A \in \arg\max_{B:|B| = c} f(B)\). We can then apply Lemma 4.6 which implies that with probability at least \((1 - k(1 - p))(1 - 1/n^2) > 1 - 4/\log n\) we have a \(1 - 1/e - 5\delta = 1 - 1/e - \epsilon\) approximation. \(\square\)

Corollary 4.15. For any monotone submodular function and any \(\epsilon > 0\), given access to a noisy oracle whose distribution has a generalized exponential tail, there is a \((1 - 1/k - \epsilon)\) approximation for \(\max_{S:|S| \leq k} f(S)\) for \(k \in \Omega(1/\epsilon)\) and \(1/4 - \epsilon\)-approximation when \(k = 1\) with probability \(1 - 4/\log n\).
Proof. Enumerate over all possible sets of size $k$ and output $\hat{A} = \arg\max f(A_{ij})$ where $A = \arg\max_{B:|B|=k} \tilde{f}(B)$. Let $A^* \in \arg\max_{B:|B|=k} f(B)$. Lemma 4.5 implies that w.h.p. for sufficiently large $n$ we have that: $\phi(A) \geq (1 - \epsilon/2)\phi(A^*)$ and from Lemma 4.2 this implies that $\phi(A) \geq (1 - \epsilon/2)(1 - 1/k)f_S(A^*)$.

Lemma 4.13 gives that $\hat{A}$ is an $1 - \epsilon/2$ approximation to $\phi(A)$ with probability $1 - 3/\log n$. 

5 Very Small $k$

The smoothing guarantee actually depends on $c \in O(1/\epsilon)$ which may not apply to small values of $k$. The dependency on $\epsilon$ originates in Claim 4.3 where we bound the variation of $c - 1$ sets $A_i - 1$, and thus the result depends on $c \geq 4/\epsilon$. For small constants the algorithm will be slightly different than the one we used before, but is rather simple and follows the same principles.

**Smoothing.** The smoothing here is straightforward. For every set $A$ consider the smoothing neighborhood $H(A) = \{ A \cup x : x \in N \setminus A \}$, $F(A) = \mathbb{E}_{X \in H(A)}[f(X)]$ and $\tilde{F}(A) = \mathbb{E}_{X \in H(A)}[\tilde{f}(X)]$.

**Lemma 5.1.** Let $A \in \arg\max_{B:|B|=k} \tilde{F}(B)$. Then, for any fixed $\epsilon > 0$ w.p. $1 - e^{-\Omega((n-k)/\epsilon)}$,

$$F(A) \geq \max_{B:|B|=k} F(A^*)$$

Proof. The proof follows the same arguments as the ones from the previous sections. Let $A^* = \arg\max_{B:|B|=k} F(B)$. We will show that w.h.p. no set $B$ for which $F(B) < (1 - \epsilon)F(A^*)$ beats $A$. The size of the smoothing set is $t = n - k$, and $\omega$ is an upper bound on the value of the noise multiplier.

Note that the optimality of $A^*$ and submodularity imply that $f(A \star \cup x) \leq 2f(A^*)$, for all $x \in N \setminus A^*$. Hence from monotonicity the variation is bounded by 2:

$$v(A^*) = \frac{\max_{x \in N \setminus A} f(A^* \cup x)}{\min_{x \in N \setminus A} f(A^* \cup x)} \leq \frac{2f(A^*)}{\min_{x \in N \setminus A} f(A^*)} = 2$$

We can therefore apply Lemma 2.5 and get that with probability at least $1 - e^{\Omega(\lambda^2 t^{1/4}/\omega)}$:

$$\tilde{F}(A^*) \geq (1 - \gamma)\mu(1 - 4t^{-1/4})F(A^*)$$

To upper bound $\tilde{F}(B)$ for a set $B$ s.t. $F(B) < (1 - \epsilon)F(A^*)$, note that the value of largest set in the smoothing neighborhood is $\max_{x \in N \setminus B} f(B \cup x) \leq 2f(A^*)$. Hence, from Lemma 2.4 we get that with probability at least $1 - e^{\Omega(\lambda^2 t^{1/4}/\omega)}$:

$$F(B) \leq (1 + \lambda)\mu(F(B) + 6t^{-1/4}F(A^*)$$

Therefore when $n$ is sufficiently large s.t. $t^{-1/4} \leq \epsilon/100$ and $\lambda < 1$ we get that:

$$F(A^*) - F(B) \geq (1 - \lambda)\mu(1 - 4t^{-1/4})F(A^*) - (1 + \lambda)\mu(F(B) + 6t^{-1/4}F(A^*))$$

$$\geq \mu\left((1 - \lambda)(1 - \frac{4\epsilon}{100})F(A^*) - (1 + \lambda)(1 - \epsilon)F(A^*) - (1 + \lambda)\frac{6\epsilon}{100}F(A^*)\right)$$

$$\geq \mu\left((1 - \lambda)(1 - \frac{4\epsilon}{100})F(A^*) - (1 + \lambda)(1 - \epsilon)F(A^*) - (1 + \lambda)\frac{6\epsilon}{100}F(A^*)\right)$$

$$> \mu \cdot F(A^*) (\epsilon - 2\lambda - 6\epsilon/5)$$

Using $\lambda < \epsilon/10$ the above inequality is strictly positive. Conditioning on the event of $\omega$ being sufficiently small completes the proof. 

\[\square\]
Approximation guarantee in expectation. The algorithm will simply select the set $\hat{A}$ to be a random set of $k$ elements from a random set of $\mathcal{H}(A)$ where $A \in \arg \max_{B:|B|=k} \tilde{F}(B)$.

**Algorithm 5** EXP-SMALL-GREEDY

**Input:** budget $k$

1. $A \leftarrow \arg \max_B \{B:|B|=k\} \tilde{F}(B)$
2. $x \leftarrow$ select random element from $N \setminus A$
3. $S \leftarrow$ random set from $A \cup x$
4. return $S$

**Theorem 5.2.** For any constant $k$ and any fixed $\epsilon > 0$ there is a $(k/(k+1) - \epsilon)$-approximation algorithm that holds in expectation.

**Proof.** From Lemma 4.2 we know that $f(\hat{A}) \geq (k/(k+1))F(A)$. Let $A^* = \arg \max_B \{B:|B|=k\} f(B)$. From monotonicity we know that $f(A^*) \leq F(A^*)$. Applying Lemma 5.1 we get that for the set $F(A) \geq (1 - \epsilon)F(A^*)$. Hence:

$$f(\hat{A}) \geq \left(\frac{k}{k+1}\right) F(A) \geq (1 - \epsilon) \left(\frac{k}{k+1}\right) F(A^*) \geq (1 - \epsilon) \left(\frac{k}{k+1}\right) f(A^*) \geq \left(\frac{k}{k+1}\right) f(A^*) - \epsilon \right) \text{OPT}$$

**High probability.** To obtain a result that holds with high probability we will do a modest variation on the algorithm above. The algorithm will enumerate all possible subsets of size $k-1$, and then select the set $\hat{A} \in \arg \max_{B:|B|=k-1} \tilde{F}(B)$. The algorithm will then select $\hat{A} \in \arg \max_{X \in \mathcal{H}(A)} \tilde{f}(X)$.

**Algorithm 6** HP-SMALL-GREEDY

**Input:** budget $k$

1. $A \leftarrow \arg \max_B \{B:|B|=k-1\} \tilde{F}(B)$
2. $S \leftarrow \arg \max_{x \in N \setminus A} \tilde{f}(A \cup x)$
3. return $S$

The analysis of the algorithm is similar to the high-probability proof from Section 4.

**Theorem 5.3.** For any constant $k$ and any fixed $\epsilon > 0$ there is a $(1 - 1/k - \epsilon)$-approximation algorithm that holds with probability at least $1 - 6/\log n$.

**Proof.** Let $A \in \arg \max_B \{B:|B|=k-1\} \tilde{F}(B)$, and let $A^* \in \arg \max_B \{B:|B|=k-1\} f(B)$. Since $A^*$ is the optimal solution over $k-1$ elements, from submodularity we know that $f(A^*) \geq (1 - 1/k)\text{OPT}$. What now remains to show is that $\hat{A} \in \arg \max_{x \in N \setminus A} \tilde{f}(A \cup x)$ is a $(1 - \epsilon)$ approximation to $F(A)$. To do so recall the definitions of good and bad sets from the previous section. Let $\delta = \epsilon/3$. Suppose that a set $X$ is in $\delta$-good($A$) if $f(X) \geq (1 - 2\delta)f(A^*)$ and in $\delta$-bad($A$) if $f(X) \leq (1 - 3\delta)f(A^*)$. We will show that the set selected has value at least as high as that of a bad set, i.e. $(1 - 3\delta)f(A^*)$ which will complete the proof.

We first show that with probability at least $1 - 6/\log n$ the noise multiplier of some good set is at least $M_g$ and of a bad set is at most $M_b$. To do so we will first argue about the size of $\delta$-good($A$) and $\delta$-bad($A$). From Lemma 5.1 and the maximality of $A$ we know that with exponentially high probability $F(A) \geq (1 - \delta)F(A^*)$. Therefore for $m = n - k$:

$$F(A) = \frac{1}{m} \sum_{x \notin A} f(A \cup x) \geq (1 - \delta) \frac{1}{m} \sum_{x \notin A^*} f(A^* \cup x) \geq (1 - \delta) f(A^*)$$
Due to the maximality of $A^*$ and submodularity we know that $f(A \cup x) \leq 2f(A^*)$ for all $x \notin A$:

$$\sum_{x \notin A} f(A \cup x) \leq |\delta\text{-good}(A)|2f(A^*) + (m - |\delta\text{-good}(A)|)(1 - 2\delta)f(A^*)$$

Putting the these bounds on $F(A)$ together and rearranging we get that:

$$|\delta\text{-good}(A)| \geq \frac{\delta \cdot m}{1 + 2\epsilon} \geq \frac{\delta m}{3}$$

Therefore, for sufficiently large $n$ the likelihood of at least one set achieving value at least $M_g$ is:

$$\Pr[\max\{\xi_1, \ldots, \xi_{\delta m/3}\} \geq M_g] \geq 1 - \left(1 - \frac{2\log n}{m}\right)^{\frac{\delta m}{1 + 2\epsilon}} \geq 1 - \frac{2}{n^{\delta/3}} \geq 1 - \frac{1}{\log n}$$

To bound $\delta\text{-bad}(A)$ we will simply note that it is trivial that $\delta\text{-bad}(A) < m$. Thus, the likelihood that all noise multipliers of bad sets are bounded from above by $M_b$ is:

$$\Pr[\max\{\xi_1, \ldots, \xi_m\} \leq M_b] \geq 1 - \left(1 - \frac{2}{m \log n}\right)^m > \left(1 - \frac{4}{\log n}\right)$$

Thus, by a union bound and conditioning on the event in Lemma 5.1 we get that $M_b$ is an upper bound on the value of the noise multiplier of bad sets and $M_g$ is with lower bound on the value of the noise multiplier of a good stem all with probability at least $1 - 6/\log n$. From Lemma 4.12 we know that for any $\gamma \in \Theta(1/\log \log n)$ we have that $M_g \geq (1 - \gamma)M_b$. Thus:

$$\max_{X \in \delta\text{-good}(A)} \tilde{f}(X) = \max_{X \in \delta\text{-good}(A)} \xi_X f(X) \geq M_g \cdot (1 - 2\delta)f(A^*) \geq (1 - \gamma)M_b \cdot (1 - 2\delta)f(A^*)$$

Let $B \in \arg \max_{C \in \delta\text{-bad}} \tilde{f}(S \cup C)$. From Claim 4.10 we know that with probability at least $1 - 2/\log n$ all noise multipliers of sets in $\epsilon\text{-bad}(A)$ are at most $M_b$. Thus:

$$\tilde{f}(S \cup B) = \max_{X \in \delta\text{-bad}} \tilde{f}(X) = \max_{X \in \epsilon\text{-bad}(A)} \xi_X f(X) \leq M_b \cdot (1 - 3\delta)f(X)$$

Putting it all together we have with probability at least $1 - 6/\log n$:

$$\tilde{f}(\hat{A}) - \tilde{f}(B) \geq M_b f(A^*) \cdot ((1 - \gamma)(1 - 2\delta) - (1 - 3\delta)) > M_b f(A^*) (\delta - \gamma)$$

Since Lemma 4.12 applies to any $\gamma \in \Theta(1/\log \log n)$, and $\delta$ is fixed it applies to $\gamma < \delta$ and the difference is positive. Since $\delta = \epsilon/6$ this completes our proof. \hfill \square

### 5.1 Information theoretic lower bounds for constant $k$

Surprisingly, even for $k = 1$ no algorithm can obtain an approximation better than $1/2$, which proves a separation between large and small $k$.\footnote{We note that if the algorithm is not allowed to query the oracle on sets of size bigger than $k$, Claim 5.5 can be extended to show an approximation ratio of $O(n)$, so choosing a random element is almost the best possible course of action.} The following is a tight bound for $k = 1$.

**Claim 5.4.** There exists a submodular function and noise distribution for which w.h.p. no randomized algorithm with a noisy oracle can obtain an approximation better than $1/2 + O(1/\sqrt{n})$ for $\max_{a \in N} f(a)$.
Indeed, since we claim that no algorithm can distinguish between the functions with probability greater than 1/2 + O(1/√n). For all sets with two or more elements, both functions return 2, and so no information is gained when querying such sets. Hence, the only information the algorithm has to work with is the number of 1, 2, and 4 values observed on singletons. If it sees the value 4 on such a set, it can conclude that the underlying function is \( f_1 \). In this case, the optimal threshold is \( \sqrt{n} \).

The probability that \( f_2 \) has at most \( \sqrt{n} \) twos is \( 1/2 - 1/\sqrt{n} \), and so is the probability that \( f_1 \) has at least \( \sqrt{n} + 1 \) twos, and hence the advantage over a random guess is \( O(1/\sqrt{n}) \) again.

An algorithm which approximates the maximal set on \( f_2 \) with ratio better than \( 1/2 + \omega(1/\sqrt{n}) \) can be used to distinguish the two functions with advantage \( \omega(1/\sqrt{n}) \). Having ruled this out, the best approximation one can get is \( 1/2 + O(1/\sqrt{n}) \) as required.

We generalize the construction to general \( k \). The lower for general \( k \) behaves like \( 2k/(2k - 1) \), where our upper bound is \( (k - 1)/k \).

**Claim 5.5.** There exists a submodular function and noise distribution for which w.h.p. no randomized algorithm with a noisy oracle can obtain an approximation better than \( (2k - 1)/2k + O(1/\sqrt{n}) \) for the optimal set of size \( k \).

**Proof.** Consider the function:

\[
\begin{align*}
    f_1(S) &= \begin{cases} 
    2|S|, & \text{if } |S| < k \\
    2k - 1, & \text{if } |S| = k \\
    2k, & \text{if } |S| > k 
    \end{cases} 
\end{align*}
\]

And the function \( f_2 \), which is dependent on the identity of some random set of size \( k \), denoted \( S^* \):

\[
\begin{align*}
    f_2(S; S^*) &= \begin{cases} 
    2|S|, & \text{if } |S| < k \\
    2k - 1, & \text{if } |S| = k, S \neq S^* \\
    2k, & \text{if } S = S^* \\
    2k, & \text{if } |S| > k 
    \end{cases} 
\end{align*}
\]

Both functions are submodular.

The noise distribution will return \( 2k/(2k - 1) \) with probability \( n^{-1/2} \) and 1 otherwise. Again we claim that no algorithm can distinguish between the functions with probability greater than 1/2. Indeed, since \( f_1, f_2 \) are identical on sets of size different than \( k \), and their value only depends on the set size, querying these sets doesn’t help the algorithm (the oracle calls on these sets can be simulated). As for sets of size \( k \), the algorithm will see a mix of \( 2k - 1, 2k \), and at most one value of \( 4k^2/(k - 1) \). If the algorithm sees the value \( 4k^2/(k - 1) \) then it was given access to \( f_2 \). However, the
algorithm will see this value only with probability $1/\sqrt{n}$. Conditioning on not seeing this value, the best policy the algorithm can adopt is to guess $f_2$ if the number of $2k$ values is at least $1 + \binom{k}{2}/\sqrt{n}$, and guess $f_1$ otherwise. The probability of success with this test is $1/2 + O(1/\sqrt{n})$ (regardless of whether the underlying function is $f_1$ or $f - 2$). Any algorithm which would approximate the best set of size $k$ to an expected ratio better than $(2k - 1)/2k + \omega(1/\sqrt{n})$ could be used to distinguish between the function with an advantage greater than $1/\sqrt{n}$, and this puts a bound of $(2k - 1)/2k + O(1/\sqrt{n})$ on the expected approximation ratio. \qed

6 Impossibility under Adversarial noise

In this section we show that there are very simple submodular functions for which no randomized algorithm with access to an $\epsilon$-erroneous oracle can obtain a reasonable approximation guarantee with a subexponential number of queries to the oracle. Intuitively, the main idea behind this result is to show that a noisy oracle can make it difficult to distinguish between two functions whose values can be very far from one another. The functions we use here are similar to those used to prove information theoretic lower bounds for submodular optimization and learning [62, 67, 30, 7, 74].

**Theorem 6.1.** No randomized algorithm can obtain an approximation strictly better than $O(n^{-1/2+\delta})$ to maximizing monotone submodular functions under a cardinality constraint using $\epsilon n^\delta/n$ queries to an $\epsilon$-erroneous oracle, for any fixed $\epsilon,\delta < 1/2$.

**Proof.** We will consider the problem of $\max_{S:|S|\leq k} f(S)$ where $k = n^{1/2+\delta}$. Let $X \subseteq N$ be a random set constructed by including every element from $N$ with probability $n^{-1/2+\delta}$. We will use this set to construct two functions that are close in expectation but whose maxima have a large gap, and show that access to a noisy oracle implies distinguishing between these two functions. The functions are:

- $f_1(S) = \min \{ |S \cap X| \cdot n^{1/2} + n^{1/2+\delta}/\epsilon, |S| \cdot n^{1+\delta} \}$
- $f_2(S) = \min \{ |S| \cdot n^{\delta} + n^{1/2+\delta}/\epsilon, |S| \cdot n^{1+\delta} \}$

By the Chernoff bound we know that $|X| \geq n^{1/2+\delta}/2$ with probability $1 - e^{-\Omega(n^{1/2+\delta})}$. Conditioned on this event we have that $\max_{S:|S|\leq k} f_1(S) = f_1(X) \in O(n^{1+\delta})$ whereas $f_2$ is symmetric and $\max_{S:|S|\leq k} f_2(S) \in O(n^{1/2+2\delta})$. Thus, an inability to distinguish between these two functions implies there is no approximation algorithm with approximation better than $O(n^{-1/2+\delta})$. We define the erroneous oracle as follows. If the function is $f_2$, its oracle returns the exact same value as $f_2$ for any given set. Otherwise, the function is $f_1$ and its erroneous oracle is defined as:

$$\tilde{f}(S) = \begin{cases} 
  f_2(S), & \text{if } (1-\epsilon)f_1(S) \leq f_2(S) \leq (1+\epsilon)f_1(S) \\
  f_1(S) & \text{otherwise}
\end{cases}$$

Notice that this oracle is $\epsilon$-erroneous, by definition.

Suppose now that the set $X$ is unknown to the algorithm, and the objective is $\max_{S:|S|\leq k} f_1(S)$. We will first show that no deterministic algorithm that uses a single query to the erroneous oracle $\tilde{f}$ can distinguish between $f_1$ and $f_2$, with exponentially high probability (equivalently, we will show that a single query the algorithm cannot find a set $S$ for which $f_1(S) < (1-\epsilon)f_2(S)$ or $f_1(S) > (1+\epsilon)f_2(S)$ with exponentially high probability). For a single query algorithm, we can imagine that the set $X$ is chosen after the algorithm chooses which query to invoke, and compute the success probability over the choice of $X$. In this case, all the elements are symmetric, and the function value is only determined by the size of the set that the single-query algorithm queries.
In case the query is a set $S$ of cardinality smaller or equal to $n^{1/2}$, by the Chernoff bound we have that $|S \cap X| \leq (1 + \beta)n^\delta$ for any $\beta < 1$ with probability at least $1 - e^{-\Omega(\beta^2 n^\delta)}$. Thus:

$$\frac{n^{1/2+\delta}}{\epsilon} \leq f_1(S) \leq \left(1 + \frac{1}{\epsilon}\right)n^{1/2+\delta}$$

$$\frac{n^{1/2+\delta}}{\epsilon} \leq f_2(S) \leq \left(1 + \frac{1}{\epsilon}\right)n^{1/2+\delta}$$

It is easy to verify that for $\beta < \epsilon/(1 - \epsilon)$: $(1 - \epsilon)f_1(S) \leq f_2(S) \leq (1 + \epsilon)f_1(S)$. Thus, for any query of size less or equal to $n^{1/2}$ the likelihood of the oracle returning $f_1$ is $1 - e^{-\Omega(n^\delta)}$.

In case the oracle queries a set of size greater than $n^{1/2}$ then again by the Chernoff bound, for any $\beta < 1$ we have that with probability at least $1 - e^{-\Omega(\beta^2 n^{1/2})}$:

$$\left(1 - \beta\right)\frac{|S|}{n^{1/2+\delta}} \leq |S \cap X| \leq \left(1 + \beta\right)\frac{|S|}{n^{1/2+\delta}}$$

For $\beta \leq \epsilon/(1 - \epsilon)$, this implies that:

$$(1 - \epsilon)f_1(S) \leq f_2(S) \leq (1 + \epsilon)f_1(S)$$

Therefore, for any fixed $\epsilon \in (0, 1)$, the algorithm cannot distinguish between $f_1$ and $f_2$ with probability $1 - e^{-\Omega(n^\delta)}$ by querying the erroneous oracle with a set larger than $n^{1/2}$. To conclude, by a union bound we get that with probability $1 - e^{-\Omega(n^\delta)}$ no algorithm can distinguish between $f_1$ and $f_2$ using a single query to the erroneous oracle, and the ratio between their maxima is $O(n^{1/2-\delta})$.

To complete the proof, suppose we had an algorithm running in time $e^{n^\delta}/n$ which can approximate the value of a submodular function, given access to an $\epsilon$-erroneous oracle with approximation ratio strictly better than $O(n^{-1/2+\delta})$ which succeeds with probability 2/3. This would let us solve the following decision problem: *Given access to an $\epsilon$-erroneous oracle for either $f_1$ or $f_2$, determine which function is being queried.* To solve the decision problem, given access to an erroneous oracle of unknown function, we would use the hypothetical approximation algorithm to estimate the value of the maximal set of size $n^{1/2+\delta}$. If this value is strictly more than $n^{1/2+2\delta}$, the function is $f_1$ (since $f_1(X) = O(n^{1+\delta})$), and otherwise it is $f_2$.

The reduction allows us to show that distinguishing between the functions in time $e^{n^\delta}/n$ and success probability 2/3 is impossible. For purpose of contradiction, suppose that there is a (randomized) algorithm for the decision problem, and let $p$ denote the probability that it outputs $f_2$ if it sees an oracle which is fully consistent with $f_2$. To succeed with probability 2/3, it must be the case that whenever the algorithm gets $f_1$ as an input, it finds a set $S$ for which the noisy oracle returns $f_1(S)$ with probability at least $2/3 - p/2 \geq 1/6$. Whenever it finds such a set, the algorithm is done, since it can compute $f_2(S)$ without calling the oracle, and hence it knows that $f_1$ was chosen in the decision problem.

In this case, we know that the algorithm makes up to $e^{n^\delta}/n$ queries, until it sees a set for which it gets $f_1(S)$. But this means that there is an algorithm with success probability at least $O(n/6e^{n^\delta})$ that makes a single query. This algorithm guesses some index $i < e^{n^\delta}/n$, and simulates the original algorithm for $i - 1$ steps (by feeding it with $f_2$ without using the oracle), and then using the oracle in step $i$. If the algorithm guesses $i$ to be the first index in which the exponential time algorithm sees $f_1(S)$, then the single query algorithm would succeed. Hence, since we showed that no single query (randomized) algorithm can find a set $S$ such that $f_1(S) < (1 - \epsilon)f_2(S)$ or $f_1(S) > (1 + \epsilon)f_2(S)$ with just one query this concludes the proof.

Somewhat surprisingly, the above theorem suggests that a good approximation to a submodular function does not suffice to obtain reasonable approximation guarantees. In particular, guarantees
from learning or sketching where the goal is to approximate a submodular function up to constant factors may not necessarily be meaningful for optimization. It is important to note that for some classes of submodular functions such as additive functions \( f(S) = \sum_{a \in S} f(a) \), we can obtain algorithms that are robust to adversarial noise. A very interesting open question is to characterize the class of submodular functions that are robust to adversarial noise.

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References


