1 Problem Statement

Given a (possibly noisy) gradient field \((p(x, y), q(x, y))\), we want to find a depth map \(Z(x, y)\) such that

\[
Z_x = \frac{\partial Z}{\partial x} = p, \quad Z_y = \frac{\partial Z}{\partial y} = q.
\] (1)

Define an error function on proposed depth \(Z\) with respect to given gradient \((p, q)\), written as \(E(Z; p, q)\). One simple and natural error function is

\[
E(Z; p, q) = (Z_x - p)^2 + (Z_y - q)^2.
\] (2)

For more general error function, we refer readers to [1]. Note that the error function \(E\) is also a map defined on every point \((x, y)\), and the problem of finding optimal \(Z(x, y)\) can be formulated as a minimization of cost function

\[
\text{cost}(Z) = \iint E(Z; p, q) dx dy.
\] (3)

2 Frankot Chellappa Algorithm [2]

2.1 General approach

Assuming the depth map can be written as a linear combination of basis function \(\phi(x, y; \omega)\), where \(\omega = (\omega_x, \omega_y) = (u, v)\) is the a 2D index

\[
Z(x, y) = \sum_{\omega} C(\omega) \phi(x, y; \omega).
\] (4)

We write the partial derivatives of basis function as

\[
\phi_x(x, y; \omega) = \frac{\partial \phi}{\partial x}(x, y; \omega), \quad \phi_y(x, y; \omega) = \frac{\partial \phi}{\partial y}(x, y; \omega),
\] (5)

and define

\[
P_x(\omega) = \iint |\phi_x(x, y; \omega)|^2 dx dy, \quad P_y(\omega) = \iint |\phi_y(x, y; \omega)|^2 dx dy.
\] (6)
Theorem 1. Given that members of \( \{ \phi_x(x, y; \omega) \}_\omega \) as well as members of \( \{ \phi_y(x, y; \omega) \}_\omega \) are mutually orthogonal, the best coefficients \( \hat{C}(\omega) \) in the (4) that minimizes the cost function (3) with square error (2) is

\[
\hat{C}(\omega) = \frac{P_x(\omega) \hat{C}_1(\omega) + P_y(\omega) \hat{C}_2(\omega)}{P_x(\omega) + P_y(\omega)},
\]

where \( \hat{C}_1(\omega) \) and \( \hat{C}_2(\omega) \) comes from expansion

\[
p(x, y) = \sum_\omega \hat{C}_1(\omega) \phi_x(x, y; \omega), \quad q(x, y) = \sum_\omega \hat{C}_2(\omega) \phi_y(x, y; \omega).
\]

2.2 Discrete Fourier basis

We use discrete Fourier basis for its computational efficiency

\[
\phi(x, y; \omega) = \exp \left( j 2\pi \left( \frac{xu}{N} + \frac{yu}{M} \right) \right),
\]

where \( M \) and \( N \) are the dimensions of the image. The partial derivatives of the basis is

\[
\phi_x(x, y; \omega) = \frac{j 2\pi u}{N} \phi(x, y; \omega), \quad \phi_y = \frac{j 2\pi v}{M} \phi(x, y; \omega),
\]

and their powers are

\[
P_x(\omega) = \left( \frac{2\pi u}{N} \right)^2, \quad P_y(\omega) = \left( \frac{2\pi v}{M} \right)^2.
\]

The expansion coefficients \( \hat{C}_1(\omega) \) and \( \hat{C}_2(\omega) \) can be calculated from the Discrete Fourier Transform (DFT) of \( p \) and \( q \), written as \( \hat{C}_p(\omega) \) and \( \hat{C}_q(\omega) \),

\[
\hat{C}_1(\omega) = -jN \hat{C}_p(\omega), \quad \hat{C}_2(\omega) = -jM \hat{C}_q(\omega).
\]

Putting everything together, the final output \( Z \) with respect to input \( p \) and \( q \) can be written as

\[
Z = \mathcal{F}^{-1} \left\{ \frac{-j 2\pi u \mathcal{F} \{ p \} + 2\pi v \mathcal{F} \{ q \}}{(\frac{2\pi u}{N})^2 + (\frac{2\pi v}{M})^2} \right\} = \mathcal{F}^{-1} \left\{ \frac{j}{2\pi} \left\{ \frac{\mathcal{F} \{ p \} + \frac{v}{M} \mathcal{F} \{ q \}}{(\frac{\pi u}{N})^2 + (\frac{\pi v}{M})^2} \right\} \right\},
\]

where \( \mathcal{F} \{ \} \) and \( \mathcal{F}^{-1} \{ \} \) are DFT and inverse DFT operations, respectively.

2.2.1 Implementation notes

1. The frequency indices \( (u, v) \) should range from \((-\lfloor N/2 \rfloor, \lfloor M/2 \rfloor)\) to \((\lfloor N/2 \rfloor, \lfloor M/2 \rfloor)\), not from \((0, 0)\) to \((M-1, N-1)\). This will affect the calculation of derivatives in (10).

2. The DC component of the depth can not be inferred from the gradient field. In the algorithm, when \( \omega = (0, 0) \), the estimated \( \hat{C}(\omega) \) will be \( \mathbb{I} \) (or \( \frac{1}{N} \)), and should be reset to a given number, say 0.

3. The Fourier basis assumes periodic depth map

\[
Z(x, 0) = Z(x, M), \quad Z(0, y) = Z(N, y).
\]

This will affect the calculation of gradient (1) at the boundary. To circumvent this restriction, we pad the surface as

\[
\bar{Z} = \begin{bmatrix} Z(x, y) & Z(N - x, y) \\ Z(x, M - y) & Z(N - x, M - y) \end{bmatrix},
\]

with corresponding gradient fields

\[
\bar{p} = \begin{bmatrix} p(x, y) & -p(N - x, y) \\ p(x, M - y) & -p(N - x, M - y) \end{bmatrix}, \quad \bar{q} = \begin{bmatrix} q(x, y) & q(N - x, y) \\ -q(x, M - y) & -q(N - x, M - y) \end{bmatrix}.
\]
References
