Math 104: Homework 6 (due March 11)

1. Determine the interior and closure of the following subsets of $\mathbb{R}$:
   
   \[ A = \{ \frac{1}{n} : n \in \mathbb{N} \}, \quad B = [0,1] \cup \mathbb{Q}. \]

2. Consider the function
   
   \[ h(x) = \begin{cases} 
   x & \text{if } x \in \mathbb{Q} \\
   0 & \text{if } x \notin \mathbb{Q}. 
   \end{cases} \]

   Show that $h$ is continuous at 0 but at no other point.

3. Ross exercise 17.9

4. In the lectures it was shown that a continuous map from $[0,1]$ to $[0,1]$ has a fixed point. Find an example of a continuous map from $(0,1)$ to $(0,1)$ which does not have a fixed point.

5. Let $f$ be a real-valued function on $\mathbb{R}$. Suppose that for a given $x \in \mathbb{R},$
   
   \[ \lim_{n \to \infty} [f(x + a_n) - f(x - a_n)] = 0 \]

   for all sequences $(a_n)$ which converge to 0. Is $f$ continuous at $x$?

6. Ross exercise 18.9

7. Ross exercise 18.10

8. Optional for the enthusiasts. A real-valued function $f$ on an interval $I$ is called convex if for all $x, y \in I$, and $0 < \lambda < 1$, then
   
   \[ f((1 - \lambda)x + \lambda y) \geq (1 - \lambda)f(x) + \lambda f(y). \]

   Suppose $f$ is convex on $[a,b]$. Prove that $f$ is continuous at $x$ for $a < x < b$, but need not be continuous at $a$ or $b$. 