Math 104: Homework 9 (due April 15)

1. Ross exercise 28.4

2. Suppose that \( f_n \) is a sequence of real-valued functions defined on an interval \([a, b]\) that converges uniformly to a function \( f \). Let \( x_0 \in [a, b] \), and suppose that

\[
\lim_{x \to x_0} f_n(x) = l_n
\]

for all \( n \in \mathbb{N} \).

(a) Prove that \( l_n \) is a Cauchy sequence, and hence that it converges to a limit \( l \).

(b) Prove that \( \lim_{x \to x_0} f(x) = l \).

3. (a) Let \( f : \mathbb{R} \to \mathbb{R} \) be a twice differentiable function, where \( f(0) = 0 \) and \( f''(x) \geq 0 \) for all \( x > 0 \). Prove that \( f(x)/x \) is increasing for \( x > 0 \).

(b) If \( f : \mathbb{R} \to \mathbb{R} \) is twice differentiable, \( f(0) = 0 \) and if \( f(x)/x \) is increasing for all \( x > 0 \), show that \( f''(x) \geq 0 \) for some \( x > 0 \), but not necessarily for all \( x > 0 \).

[Hint: consider \( f(x) = x(1 - e^{-x}) \).]

4. Ross exercise 29.17

5. Let \( f(x) = |x|^3 \). Compute \( f'(x), f''(x) \), and show that \( f^{(3)}(0) \) does not exist.

6. Ross exercise 30.1

7. Suppose that \( f \) is differentiable at a point \( a \). Define

\[
L_1(a, h) = \frac{f(a + h) - f(a - h)}{2h},
\]

\[
L_2(a, h) = \frac{-f(a + 2h) + 8f(a + h) - 8f(a - h) + f(a - 2h)}{12h}.
\]

(a) Prove that \( \lim_{h \to 0} L_i(a, h) = f'(a) \) for \( i = 1, 2 \).

(b) Consider the case when \( f(x) = x^5 \). How does \( |L_i(a, h) - f'(a)| \) behave as \( h \to 0 \) for \( i = 1, 2, 3 \)? Is there a difference in the rate of convergence?