Math 104: Homework 10 (due April 20)

1. (a) Construct a Taylor series expansion for the function \( f(x) = \log(1 + x) \) at \( x = 0 \). You can assume basic properties of logarithm.

(b) Use Taylor’s theorem to write down an expression for the remainder \( R_n(x) \), and use this to prove that the Taylor series agrees with \( f \) in the range \( x \in (-1/2, 1) \). Note: it can be shown that the Taylor series agrees with \( f \) for \( x \in (-1, 1) \), as discussed in Chapter 26 of Ross. However, here, to illustrate Taylor’s theorem, only a subset of this interval is considered.

2. Suppose \( f \) is a continuous function on \([a, b]\), and \( f(x) \geq 0 \) for all \( x \in [a, b] \). Prove that if \( \int_a^b f = 0 \), then \( f(x) = 0 \) for all \( x \in [a, b] \).

3. Construct an example of a function where \( f(x)^2 \) is integrable on \([0, 1]\) but \( f(x) \) is not.

4. Ross exercise 33.7

5. Ross exercise 33.8

6. (a) For any two numbers \( u, v \in \mathbb{R} \), prove that \( uv \leq (u^2 + v^2)/2 \). Let \( f \) and \( g \) be two integrable functions on \([a, b]\). Prove that if \( \int_a^b f^2 = 1 \) and \( \int_a^b g^2 = 1 \) then

\[
\int_a^b fg \leq 1.
\]

(b) Prove the Schwarz inequality, that for any two integrable functions \( f \) and \( g \) on an interval \([a, b]\),

\[
\left| \int_a^b fg \right| \leq \left( \int_a^b f^2 \right)^{1/2} \left( \int_a^b g^2 \right)^{1/2}.
\]

(c) Let \( X \) be the set of all continuous functions on the interval \([a, b]\). For any \( f, g \in X \), define

\[
d(f, g) = \left( \int_a^b |f - g|^2 \right)^{1/2}.
\]

Prove that \( d \) is a metric.