Math 104: Homework 11 (due April 27)

1. Ross exercise 34.3
2. Ross exercise 34.12
3. Ross exercise 36.6
4. Find a sequence of integrable functions \((f_n)\) on \(\mathbb{R}\) where \(\int_{-\infty}^{\infty} f_n = 1\) for all \(n\), but \(f_n \to 0\) uniformly on \(\mathbb{R}\).

5. (a) By using simple properties of \(\sin x\) and \(\cos x\), show how to define the function 
\[ \tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}. \]
Prove that it is differentiable, strictly increasing, and neither bounded above nor below.

(b) By using inverse function theorems, define \(\tan^{-1} : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\) and show that 
\[ (\tan^{-1})'(x) = \frac{1}{1+x^2}. \]

(c) Prove that for \(|x| < 1\), 
\[ \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}. \]

(d) By making use of Abel’s theorem, or otherwise, show that 
\[ \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}. \]

(e) **Optional for the enthusiasts.** Calculate \((5+i)^4(239-i)\) and use it to prove Machin’s formula 
\[ \frac{\pi}{4} = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}. \]

6. Let \(I_n = \int_{0}^{\pi/2} \sin^n x \, dx\).

(a) Prove that \(I_0 = \frac{\pi}{2}\) and \(I_1 = 1\).

(b) Use integration by parts to prove that \((n+1)I_n = (n+2)I_{n+2}\) for all \(n \geq 0\).

(c) Prove that \(I_{2m+1} \leq I_{2m} \leq (1 + \frac{1}{2m})I_{2m+1}\) for all \(m \in \mathbb{N}\), and hence that \(I_{2m}/I_{2m+1} \to 1\) as \(m \to \infty\).

(d) Prove that for \(m \in \mathbb{N}\), 
\[ \frac{\pi}{2} = \frac{246}{135} \cdots \frac{2m}{2m-1} I_{2m}, \quad 1 = \frac{357}{246} \cdots \frac{2m+1}{2m} I_{2m+1}, \]
and hence that 
\[ \frac{\pi}{2} = \lim_{m \to \infty} \frac{24466}{133557} \cdots \frac{2m}{2m-12m+1}. \]