Math 104: Homework 5 (due March 2)

1. (a) By using the integral test, or otherwise, prove that
\[ \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \]
converges if and only if \( p > 1 \).
(b) **Optional for the enthusiasts.** Suppose \((a_n)\) is a non-increasing sequence of positive real numbers, and that \( \sum a_n \) converges. By considering the Cauchy criterion, or otherwise, prove that \( na_n \to 0 \) as \( n \to \infty \). By considering part (a), show that the converse result is not true.

2. Consider the following functions defined for \( x, y \in \mathbb{R} \):
\[ d_1(x, y) = (x - y)^2, \quad d_2(x, y) = \sqrt{|x - y|}, \quad d_3(x, y) = |x^2 - y^2|, \quad d_4(x, y) = |x - 2y| \]
For each function, determine whether it is a metric or not.

3. Ross exercise 13.3

4. Consider two-dimensional space \( \mathbb{R}^2 \), with positions written as \( u = (u_1, u_2) \), and the Euclidean norm defined as \( ||u|| = (u_1^2 + u_2^2)^{1/2} \). The Poincaré disk model consists of the points \( S = \{ u : ||u|| < 1 \} \), with metric
\[ d(u, v) = \cosh^{-1} \left[ 1 + \frac{2||u - v||^2}{(1-||u||^2)(1-||v||^2)} \right] \]
for all \( u, v \in S \). Define \( r = \cosh^{-1} 1/4 \). Draw the Poincaré disk, and then calculate and draw the neighborhoods \( N_r(u) \) for \( u = (0,0) \), \((1/2,0)\), and \((3/4,0)\). [This can be done analytically, although if you prefer, you can also make use of a computer.]

5. Suppose that \( d_1 \) and \( d_2 \) are equivalent metrics for a set \( S \). Prove that if a sequence \( (s_n) \) converges to \( s \) with respect to \( d_1 \), then it also converges with respect to \( d_2 \).

6. Suppose that \( (p_n) \) is a Cauchy sequence in a set \( S \) with metric \( d \), and that some subsequence \( (p_{nk}) \) converges to a point \( p \in S \). Prove that the full sequence \( (p_n) \) converges to \( p \).

7. Suppose that \( (p_n) \) and \( (q_n) \) are Cauchy sequences in a set \( S \) with metric \( d \). Define \( (a_n) = d(p_n, q_n) \). Show that the sequence \( (a_n) \) converges. It may be useful to consider the triangle inequality
\[ d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) \]
which is true for all \( n \) and \( m \).

8. **Optional for the enthusiasts.** Consider two-dimensional space \( \mathbb{R}^2 \) as in Exercise 4. Define an alternative norm as \( ||u||_S = (u_1^2 + u_2^2 + u_1 u_2)^{1/2} \). Prove that the function \( d_S(u, v) = ||u - v||_S \) is a metric, and that it is equivalent to the Euclidean metric.