Math 104: Homework 10 solutions

1. (a) The first derivative is

\[ f'(x) = \frac{1}{x+1} \]

and the \( n \)th derivative is given by

\[ f^{(n)} = \frac{(n-1)!(-1)^{n-1}}{(x+1)^n}. \]

Hence the Taylor series expansion at \( x = 0 \) is given by

\[ f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{n!} + R_n(x) \]

\[ = \sum_{k=1}^{n-1} \frac{(n-1)!}{n!} (-1)^{n-1} + R_n(x) \]

\[ = \sum_{k=1}^{n-1} \frac{(-1)^{n-1}}{n} + R_n(x). \]

(b) By using Taylor’s Theorem, the remainder is given by

\[ R_n(x) = \frac{f^{(n)}(y)}{n!} x^n = \frac{(-1)^{n-1}}{n(y+1)^n} x^n = \frac{(-1)^{n-1}}{n} \left( \frac{x}{1+y} \right)^n \]

where \( y \) is between 0 and \( x \). If \( 0 < x < 1 \), then \( y > 0 \), so \( 0 < x/(1+y) < x \) and hence \( R_n(x) \to 0 \) as \( n \to 0 \). If \( -1/2 < x < 0 \), then \( 1+y > 1/2 \), so \( x/(1+y) < (1/2)2 = 1 \) and hence \( R_n(x) \to 0 \) as \( n \to 0 \). Hence the Taylor series agrees with \( f \) in the range \(-1/2 < x < 1\).

2. Suppose that there exists a \( y \in [a,b] \) such that \( f(y) > 0 \). Suppose \( y = a \); then since \( f \) is continuous, there exists a \( \delta > 0 \) such that \( |x-a| < \delta \) implies that \( |f(a) - f(x)| < f(a)/2 \), in which case \( f(a + \delta/2) > 0 \). Similarly, if \( y = b \), then there exists a \( \delta > 0 \) such that \( f(b - \delta/2) > 0 \). Hence there must exist an \( x_0 \in (a,b) \) such that \( f(x_0) > 0 \). Since \( f \) is continuous, there exists a \( \delta > 0 \) such that \( |x - x_0| < \delta \) implies that

\[ |f(x) - f(x_0)| < \frac{f(x_0)}{2} \]

and hence

\[ f(x) > \frac{f(x_0)}{2}. \]
Since \( x_0 \in (a, b) \) it is always possible to find a \( \delta > 0 \) to that \( (x_0 - \delta, x_0 + \delta) \subset [a, b] \).
Consider the partition \( P = \{a = t_0 < t_1 < t_2 < t_3 = b\} \) where \( t_1 = x_0 - \frac{\delta}{2} \) and \( t_2 = x_0 + \frac{\delta}{2} \). Then
\[
L(f, P) = \sum_{k=1}^{3} (t_k - t_{k-1})m(f, [t_{k-1}, t_k])
\]
Since \( f(x) \geq 0 \) for all \( x \in [a, b] \), then \( m(f, [a, t_1]) \geq 0 \) and \( m(f, [t_2, b]) \geq 0 \). In addition, by reference to Eq.\[1\] \( m(f, [t_1, t_2]) \geq f(x_0)/2 \), and hence
\[
L(f, P) \geq 0 + \left(x_0 + \frac{\delta}{2} - x_0 + \frac{\delta}{2}\right) \frac{f(x_0)}{2} + 0 = \frac{\delta x_0}{2} > 0.
\]
Hence \( \int_{a}^{b} f > 0 \), which is a contradiction. Hence, \( f(x) = 0 \) for all \( x \in [a, b] \).

3. Consider the function
\[
f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\
-1 & \text{if } x \notin \mathbb{Q}
\end{cases}
\]
defined on \([0, 1]\). Then \( f(x)^2 = 1 \) for all \( x \in [0, 1]\), and this function is integrable.
Consider any partition \( P = \{0 = t_0 < t_1 < \ldots < t_n = 1\} \). Then \( m(f, [t_{k-1}, t_k]) = -1 \) for \( k = 1, \ldots, n \) since any interval of finite length must contain an irrational number. Similarly, \( M(f, [t_{k-1}, t_k]) = 1 \) for \( k = 1, \ldots, n \) since any interval of finite length must also contain a rational number. Thus
\[
L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}) = - \sum_{k=1}^{n} (t_k - t_{k-1}) = -(1 - 0) = -1
\]
and
\[
U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} (t_k - t_{k-1}) = 1 - 0 = 1.
\]
Since this is true for any partition, it follows that \( U(f) = 1 \), and \( L(f) = -1 \), so \( f \) is not integrable.

4. (a) Let \( f \) be a bounded function on \([a, b]\), so that there exists a \( B > 0 \) such that \( |f(x)| \leq B \) for all \( x \in [a, b] \). Recall the result from Ross Exercise 4.14, that for two sets \( A \) and \( B \), then the set \( C \) of sums \( a + b \) where \( a \in A \) and \( b \in B \) satisfies
\[
\sup C = \sup A + \sup B.
\]
For any interval \( I \subseteq [a, b] \),
\[
M(f^2, I) - m(f^2, I) = \sup \{f(x)^2 : x \in I \} - \inf \{f(x)^2 : x \in I \}
= \sup \{f(x)^2 : x \in I \} + \sup \{-f(x)^2 : x \in I \}
= \sup \{f(x)^2 - f(y)^2 : x, y \in I \}
= \sup \{(f(x) - f(y))(f(x) + f(y)) : x, y \in I \}
\leq 2B \sup \{f(x) - f(y) : x, y \in I \}
= 2B (\sup \{f(x) : x \in I \} - \inf \{f(x) : x \in I \})
\]
Now consider any partition \( P = \{ a = t_0 < t_1 < \ldots < t_n = b \} \). Then

\[
U(f^2, P) - L(f^2, P) = \sum_{k=1}^{n} (t_k - t_{k-1})(M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k]))
\]

\[
\leq 2B \sum_{k=1}^{n} (t_k - t_{k-1})(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]))
\]

\[
= 2B[U(f, P) - L(f, P)].
\]

(b) If \( f \) is integrable, then for all \( \epsilon > 0 \) there exists a partition \( P \) such that

\[
U(f, P) - L(f, P) < \frac{\epsilon}{2B}.
\]

Thus by using the above inequality,

\[
U(f^2, P) - L(f^2, P) \leq 2B(U(f, P) - L(f, P)) < \epsilon
\]

and hence \( f^2 \) is integrable.

5. (a) Since \( f \) and \( g \) is integrable, then \( f + g \) and \( f - g \) are integrable by Theorem 33.3. The result from the previous question shows that \( (f + g)^2 \) and \( (f - g)^2 \) are integrable also. Applying Theorem 33.3 again shows that

\[
f g = \frac{(f + g)^2 - (f - g)^2}{4}
\]

is integrable.

(b) By Theorem 33.5, \( |f - g| \) is integrable, and thus by Theorem 33.3,

\[
\max(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|
\]

is integrable. Since \( -f \), and \( -g \) are integrable,

\[
\min(f, g) = -\max(-f, -g)
\]

is integrable also.

6. (a) For any two numbers \( u, v \in \mathbb{R} \),

\[
(u + v)^2 \geq 0
\]

so

\[
u^2 + 2uv + v^2 \geq 0
\]
and hence $uv \leq (u^2 + v^2)/2$. Consider two integrable functions $f$ and $g$ on $[a, b]$, where $\int_a^b f^2 = \int_a^b g^2 = 1$. Then by Exercise 33.8, $fg$ is integrable. Since $f(x)g(x) \leq (f(x)^2 + g(x)^2)/2$, Theorem 33.4 shows that

$$\int_a^b fg \leq \int_a^b \frac{f^2 + g^2}{2}$$

and hence

$$\int_a^b fg \leq \left(\int_a^b \frac{f^2}{2}\right) + \left(\int_a^b \frac{g^2}{2}\right) = \frac{1}{2} + \frac{1}{2} = 1.$$

(b) Consider two integrable functions $f$ and $g$ on $[a, b]$. Define $C = \int_a^b f^2$ and $D = \int_a^b g^2$. Suppose initially that both $C > 0$ and $D > 0$, then define $F(x) = f(x)/\sqrt{C}$ and $G(x) = g(x)/\sqrt{D}$ for $x \in [a, b]$. Hence

$$\int_a^b F^2 = \int_a^b \frac{f^2}{C} = \frac{1}{C} \int_a^b f^2 = 1$$

and similarly $\int_a^b G^2 = 1$, so the inequality of the previous section can be applied to show that

$$\int_a^b FG \leq 1.$$ 

Thus

$$\int_a^b \frac{fg}{CD} \leq 1$$

so

$$\int_a^b fg \leq CD = \left(\int_a^b f^2\right)^{1/2} \left(\int_a^b g^2\right)^{1/2}.$$ 

Since the same inequality would hold if applied to $-f$ and $g$, it follows that

$$\left|\int_a^b fg\right| \leq \left(\int_a^b f^2\right)^{1/2} \left(\int_a^b g^2\right)^{1/2}.$$ 

Now consider the case when $C = 0$ or $D = 0$. If $f$ and $g$ are continuous, then the result follows quickly from question 3. If $C = 0$, then $f(x)^2 = 0$ for all $x \in [a, b]$, and hence $f(x) = 0$ for all $x \in [a, b]$, in which case the Schwarz inequality is satisfied. Similarly, if $D = 0$, then $g(x) = 0$ for all $x \in [a, b]$, and the Schwarz inequality is satisfied.

However, if $f$ and $g$ are not assumed to be continuous, the result requires a more direct approach. Suppose $C = 0$ and $D \neq 0$, and consider any $\epsilon > 0$. Then there exists a partition $P$ such that

$$U(f^2, P) - L(f^2, P) < \epsilon.$$
It is known that $L(f^2, P) \geq 0$ since $f(x)^2 \geq 0$ for all $x$. Since $L(f^2, P) \leq \int_a^b f^2 = 0$, it follows that $L(f^2, P) = 0$. If $U(f^2, P) = 0$, then that implies that $f(x) = 0$ for all $x \in [a, b]$, and hence the Schwarz inequality is satisfied. Otherwise, consider the step function

$$h_P(x) = \begin{cases} M(|f|, [t_{k-1}, t_k]) & \text{for } x \in [t_{k-1}, t_k), \\ M(|f|, [t_{n-1}, b]) & \text{for } x = b. \end{cases}$$

By construction,

$$\int_a^b h_P^2 = U(f^2, P).$$

and since $U(f^2, P) > 0$, the Schwarz inequality can be applied to $h_P$ and $g$, to show that

$$\left| \int_a^b fg \right| \leq \int_a^b |fg| \leq \int_a^b |h_P g| \leq \left( \int_a^b h_P^2 \right)^{1/2} \left( \int_a^b g^2 \right)^{1/2} = \left( U(f^2, P) \right)^{1/2} \cdot D < D\sqrt{\epsilon}.$$ 

Since this is true for arbitrary $\epsilon > 0$, it follows that $|\int_a^b fg| = 0$, and thus the Schwarz inequality is satisfied. The same argument can then be repeated to show that the result also holds for $D = 0$. Hence the Schwarz inequality is satisfied for all functions.

(c) Consider the three properties of a metric:

M1. Consider $f \in X$. Then

$$d(f, f) = \left( \int_a^b |f - f|^2 \right)^{1/2} = \left( \int_a^b 0 \right)^{1/2} = 0.$$ 

Suppose $f, g \in X$ and $d(f, g) = 0$. Then

$$0 = \int_a^b |f - g|^2$$

and by the result from question 3, since $|f - g|^2$ is continuous, it follows that $|f(x) - g(x)|^2 = 0$ for all $x \in [a, b]$. Hence $f(x) = g(x)$ for all $x \in [a, b]$. Thus $d(f, g) = 0$ if and only if $f = g$. 

M2. For any two $f, g \in X$,

$$d(f, g) = \left( \int_a^b |f - g|^2 \right)^{1/2} = \left( \int_a^b |g - f|^2 \right)^{1/2} = d(g, f)$$

so $d$ is symmetric.

M3. Consider $f, g, h \in X$. If $d(f, h) = 0$, then the inequality $d(f, h) \leq d(f, g) + d(g, h)$ is immediately satisfied. Otherwise consider

$$d(f, h)(d(f, g) + d(g, h)) = \left( \int_a^b |f - g|^2 \right)^{1/2} \left( \int_a^b |f - h|^2 \right)^{1/2}$$

$$+ \left( \int_a^b |g - h|^2 \right)^{1/2} \left( \int_a^b |f - h|^2 \right)^{1/2} \geq \left| \int_a^b (f - g) \cdot (f - h) \right| + \left| \int_a^b (g - h) \cdot (f - h) \right|$$

$$= \int_a^b |f - g| \cdot |f - h| + \int_a^b |g - h| \cdot |f - h|$$

$$= \int_a^b (|f - g| + |g - h|) \cdot |f - h|.$$  

The usual triangle inequality can be applied to show that

$$d(f, h)(d(f, g) + d(g, h)) \geq \int_a^b |f - h| \cdot |f - h| = d(f, h)^2$$

and hence, since $d(f, h) > 0$,

$$d(f, g) + d(g, h) \geq d(f, h)$$

so the triangle inequality is satisfied.

Hence $d$ is a metric.