Math 104: Homework 2 solutions

1. \( A = (0, \infty) \): Since this is an open interval, the minimum is undefined, and since the set is not bounded above, the maximum is also undefined. \( \inf A = 0 \) and \( \sup A = \infty \).

\( B = \left\{ \frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N} \right\} \): This set does not have a minimum, since for any element \( \frac{1}{m} + \frac{1}{n} \), there is a smaller element \( \frac{1}{m+1} + \frac{1}{n} \). The maximum element is 2, which is attained for \( m = n = 1 \). Hence \( \max B = 2 \). \( B \) is a bounded below by 0. However, for \( \epsilon > 0 \), there exists \( c \in \mathbb{N} \) such that \( 1/c < \epsilon \), by the Archimedean property. Thus by putting \( m = n = 2c \), we see that there exists \( b \in B \) such that \( b < \epsilon \). Hence, \( \epsilon \) is not a lower bound. Hence 0 is the greatest lower bound, and thus \( \inf B = 0 \).

\( C = \{ x^2 - x - 1 : x \in \mathbb{R} \} \): By completing the square, this can be written as \( \{ (x - \frac{1}{2})^2 - \frac{5}{4} | x \in \mathbb{R} \} \). We know that \( (x - \frac{1}{2})^2 \geq 0 \) by Theorem 3.2(iv). Hence \( \min C = -\frac{5}{4} \) which is attained for \( x = \frac{1}{2} \). Since the values of \( C \) are not bounded above, the maximum does not exist. Hence \( \inf C = -\frac{5}{4} \) and \( \sup C = \infty \).

\( D = [0, 1] \cup [2, 3] \): Since this set is composed of closed intervals, we have \( \min D = 0 \) and \( \max D = 3 \). Hence \( \inf D = 0 \) and \( \max D = 3 \), and thus \( \inf D = 0 \) and \( \sup D = 3 \).

\( E = \bigcup_{n=1}^{\infty} [2n, 2n+1] \): The first interval in this union is \( [2, 3] \), and there are an infinite number of consecutive intervals in the positive direction. Hence \( \min E = 2 \), but the maximum does not exist. Thus \( \inf E = 2 \) and \( \sup E = \infty \).

\( F = \cap_{n=1}^{\infty} (1 - \frac{1}{n}, 1 + \frac{1}{n}) \): We begin by showing that \( F = \{1\} \). Choose any \( x > 1 \). Then \( x = 1 + \epsilon \) for \( \epsilon > 0 \). Hence, by the Archimedean property, there exists \( n \in \mathbb{N} \) such that \( \frac{1}{n} < \epsilon \). Hence \( x \notin (1 - \frac{1}{n}, 1 + \frac{1}{n}) \), and thus \( x \notin F \). Similarly if \( x < 1 \), then \( x = 1 - \epsilon \), and and there exists an \( n \) such that \( x \notin (1 - \frac{1}{n}, 1 + \frac{1}{n}) \).

However, \( 1 \in (1 - \frac{1}{n}, 1 + \frac{1}{n}) \) for all \( n \in \mathbb{N} \). Thus \( F = \{1\} \), and hence \( \min F = \max F = \inf F = \sup F = 1 \).

2. (a) To begin, we show that \( \sup A + \sup B \) is an upper bound for \( S \). Any element in \( S \) can be written as \( a + b \) for \( a \in A \), and \( b \in B \). However, since \( \sup A \) is an upper bound for \( A \), then \( a \leq \sup A \). Similarly, \( b \leq \sup B \), and thus \( a + b \leq \sup A + \sup B \).

We now wish to show that \( \sup A + \sup B \) is the least upper bound for \( S \). Assume that \( t \) is a upper bound for \( S \), but that \( t < \sup A + \sup B \). Then for some \( \epsilon > 0 \), \( t = \sup A + \sup B - \epsilon \). Now, since \( \sup A \) is the supremum of \( A \), there exists \( a \in A \) such that \( a > \sup A - \frac{\epsilon}{2} \). (If this was not the case, then \( \sup A - \frac{\epsilon}{2} \) would be an upper bound for \( A \).) Similarly, there exists \( b \in B \) such that \( b > \sup B - \frac{\epsilon}{2} \). But \( a + b \in S \), and

\[
a + b > \left( \sup A - \frac{\epsilon}{2} \right) + \left( \sup B - \frac{\epsilon}{2} \right) = t.
\]
Hence \( t \) is not an upper bound, which is a contradiction. Thus if \( t \) is an upper bound, it must satisfy \( t \geq \sup A + \sup B \).

\( \sup A + \sup B \) is an upper bound for \( S \), and it is the least upper bound. Hence \( \sup S = \sup A + \sup B \).

(b) This could be proved by repeating the above argument but with lower bounds instead of upper bounds. However, an alternative method is to define negated sets \(-A = \{-a|a \in A\}, -B = \{-b|b \in B\}, \) and \(-S = \{-s|s \in S\}\).

We see that \(-S\) can be constructed as the set of sums \( a' + b' \) where \( a' \in -A \) and \( b' \in -B \). Thus, by applying the above result, we know that \( \sup(-S) = \sup(-A) + \sup(-B) \). However, by Corollary 4.5, for any set \( C \), \( \sup(-C) = -\inf C \). Hence \( -\inf S = -\inf A - \inf B \) and thus \( \inf S = \inf A + \inf B \).

3. This result is not true. As a counterexample, choose \( A = B = \{-2, 1\} \). Then \( \sup A = \sup B = 1 \), and hence \( \sup A \cdot \sup B = 1 \). However \( M = \{-2, 1, 4\} \) and hence \( \sup M = 4 \) which is not equal to 1.

Note that the counterexample relies on having two negative terms that multiply together to give a large positive term. If we restrict \( A \) and \( B \) to be subsets of the positive real line, \((0, \infty)\), then the result \( \sup M = \sup A \cdot \sup B \) would hold, and could be proved following similar logic to the previous exercise.

4. (a) By dividing through by \( n \), we obtain

\[
\left( \frac{3n}{n+3} \right)^2 = \left( \frac{3}{1 + \frac{3}{n}} \right)^2
\]

and since \( \frac{1}{n} \to 0 \) as \( n \to \infty \), we see that

\[
\left( \frac{3n}{n+3} \right)^2 \to \left( \frac{3}{1} \right)^2 = 9.
\]

(b) By making use of Example 1 in Section 1, we can write

\[
\frac{1 + 2 + \ldots + n}{n^2} = \frac{n(n+1)/2}{n^2} = \frac{n+1}{2n} = \frac{1+\frac{1}{n}}{2} \to \frac{1}{2}
\]

as \( n \to \infty \).
(c) We first write
\[
\frac{a^n - b^n}{a^n + b^n} = \frac{1 - \left(\frac{b}{a}\right)^n}{1 + \left(\frac{b}{a}\right)^n} = \frac{1 - c^n}{1 + c^n}
\]
where \(c = b/a\). Since \(a > b > 0\), we know that \(1 > c > 0\). Thus \(c^n \to 0\) as \(n \to \infty\) by Theorem 9.7(b), and hence \((a^n - b^n)/(a^n + b^n) \to 1\) as \(n \to \infty\).

(d) Although \(2^n\) rapidly becomes much bigger than \(n^2\), we must be careful to show this rigorously. One method is to use the binomial theorem to expand for \(n \geq 3\) according to
\[
2^n = (1 + 1)^n = 1^n + n \cdot 1^{n-1} \cdot 1 + \frac{n(n-1)}{2} \cdot 1^{n-2} \cdot 1^2 + \frac{n(n-1)(n-2)}{6} \cdot 1^{n-3}1^3 + \ldots
\]
and hence, by neglecting all but one term,
\[
2^n > \frac{n(n-1)(n-2)}{6}.
\]
Now, for \(n \geq 3\), we know that \((n-1) > \frac{n}{2}\) and \((n-2) > \frac{n}{4}\), and hence
\[
2^n > \frac{n^3}{24}
\]
and therefore \(2^n > n^3/24\). Thus for \(n \geq 3\), \(0 < n^2/2^n < 24/n\), and thus \(n^2/2^n \to 0\) as \(n \to \infty\) by the Squeezing Lemma.

(e) This can be carried out by introducing a factor that completes the square:
\[
\sqrt{n + 1} - \sqrt{n} = \left(\sqrt{n + 1} - \sqrt{n}\right) \frac{\sqrt{n + 1} + \sqrt{n}}{\sqrt{n + 1} + \sqrt{n}}
\]
\[
= \left(\sqrt{n + 1} - \sqrt{n}\right) \frac{\sqrt{n + 1} + \sqrt{n}}{\sqrt{n + 1} + \sqrt{n}}
\]
\[
= \frac{(n + 1) - n}{\sqrt{n + 1} + \sqrt{n}}
\]
\[
= \frac{1}{\sqrt{n + 1} + \sqrt{n}}.
\]
Since \(\sqrt{n} \to \infty\) as \(n \to \infty\), we must have \(\sqrt{n + 1} - \sqrt{n} \to 0\) as \(n \to \infty\).

5. (a) Let \(s_n = \frac{\sqrt{2}}{n}\) for all \(n \in \mathbb{N}\). We know that \(s_n\) is irrational, since if \(s_n = p/q\) for some integers \(p\) and \(q\), then \(\sqrt{2} = p/(qn)\), but \(\sqrt{2}\) has been shown to be irrational. Now consider an \(\epsilon > 0\). We see that
\[
|s_n - 0| = \frac{\sqrt{2}}{n}
\]
and thus if \(n > \sqrt{2}\epsilon\), then \(|s_n - 0| < \epsilon\). Hence \(s_n \to 0\) as \(n \to \infty\).
There are many ways this could be achieved, such as defining $s_n$ as the first digits of $\pi$, so that the first few terms would be $3, 3.1, 3.14, 3.141, 3.1415, 3.14159$. However, here a method is presented which shows explicitly how to construct all the numbers in a sequence, and show that they converge to an irrational.

Define $s_n = p_n/q_n$ and put $p_1 = 1$ and $q_1 = 1$. Now, define the rest of the sequence recursively by putting

$$p_{n+1} = p_n + 2q_n, \quad q_{n+1} = p_n + q_n.$$  

It is straightforward to see that if $p_n > 0$ and $q_n > 0$, then $p_{n+1} > 0$ and $q_{n+1} > 0$, so by mathematical induction $s_n > 0$ and $q_n \neq 0$ for all $n$. The first few terms are


The last of these is $1.4142156862 \ldots$ which differs from $\sqrt{2}$ by $2.12 \times 10^{-6}$. Here, we prove that $s_n$ does indeed converge to $\sqrt{2}$. Suppose that $s_n$ differs from $\sqrt{2}$ by an amount $\Delta_n$, so that

$$\frac{p_n}{q_n} - \sqrt{2} = \Delta_n.$$  

Then consider how much $s_{n+1}$ differs from $\sqrt{2}$:

$$\Delta_{n+1} = \frac{p_{n+1}}{q_{n+1}} - \sqrt{2}$$  

$$= \frac{p_n + 2q_n}{p_n + q_n} - \sqrt{2}$$  

$$= \frac{\frac{p_n}{q_n} + 2}{\frac{p_n}{q_n} + 1} - \sqrt{2}$$  

$$= \frac{\Delta_n + \sqrt{2} + 2}{\Delta_n + \sqrt{2} + 1} - \sqrt{2}$$  

$$= \frac{\Delta_n + \sqrt{2} + 2 - \sqrt{2}(\Delta_n + \sqrt{2} + 1)}{\Delta_n + \sqrt{2} + 1}$$  

$$= \frac{(1 - \sqrt{2})\Delta_n}{\Delta_n + \sqrt{2} + 1}.$$  

Since $p_n/q_n$ is positive, we know from Eq. 1 that $\Delta_n + \sqrt{2} > 0$. We also know that $1 < \sqrt{2} < 3/2$ since $1^2 = 1 < 2$ and $(3/2)^2 = 9/4 > 2$. Hence $-1/2 <$
1 − √2 < 0. Using these inequalities,

\[ |\Delta_{n+1}| = |\Delta_n| \cdot \left| \frac{1 - \sqrt{2}}{\Delta_n + \sqrt{2} + 1} \right| \]

\[ \leq |\Delta_n| \cdot \left| \frac{1/2}{1} \right| \]

\[ \leq \frac{|\Delta_n|}{2}. \]

Hence by mathematical induction, \( |\Delta_n| \leq |\Delta_1|(1/2)^{n-1} \), and thus by Theorem 9.7(b), \( |\Delta_n| \to 0 \) as \( n \to \infty \). Hence Eq. 1 shows that \( s_n = p_n/q_n \to \sqrt{2} \) as \( n \to \infty \).

6. We can rewrite a term in the sequence as a product of fractions,

\[ s_n = \left( \frac{1}{n} \right) \left( \frac{2}{n} \right) \ldots \left( \frac{n}{n} \right). \]

Each of these fractions is less than or equal to one, and the first is equal to \( \frac{1}{n} \). Thus \( s_n \leq \frac{1}{n} \). Now choose \( \epsilon > 0 \). We see that

\[ |s_n - 0| = s_n \leq \frac{1}{n} \]

and thus we see that for all \( n > \epsilon^{-1} \), \( |s_n - 0| < \epsilon \). Hence \( \lim_{n \to \infty} s_n = 0 \).

7. An arbitrary polynomial can be written as a sum

\[ p(x) = \sum_{j=0}^{k} a_j x^j \]

where \( a_j \in \mathbb{R} \) and \( a_k \neq 0 \). To begin, we show by induction that if \( s_n \to s \) as \( n \to 0 \), then \( (s_n)^j \to s^j \) for all \( j \in \mathbb{N} \cup \{0\} \). Consider the base case when \( j = 0 \). Since \( (s_n)^0 = 1 \) for all \( n \), this is a constant sequence, and thus converges to 1, which is equal to \( s^0 \).

Now assume the result is true for \( j \) and consider the case for \( j + 1 \). We can define \( (s_n)^{j+1} = (s_n)^j \cdot s_n \) and thus by Theorem 9.4, we know that \( (s_n)^j \cdot s_n \to s^j \cdot s = s^{j+1} \) as \( n \to \infty \). Hence the induction step holds, and by mathematical induction \( (s_n)^j \to s^j \) for all \( j \in \mathbb{N} \cup \{0\} \).

Now, if \( a_j \) is a constant, then we know that \( a_j (s_n)^j \to a_j s^j \) by Theorem 9.2. Finally, by applying Theorem 9.3, we see that \( p(s_n) \to p(s) \) as \( n \to \infty \).
8. We begin by proving that \( s_n = f(n) \) which is defined according to 

\[
f(n) = 2 - (2 - t)2^{1-n}.
\]

Consider the case when \( n = 1 \):

\[
f(1) = 2 - (2 - t)2^{1-1} = 2 - (2 - t) = t
\]

and thus \( s_1 = f(1) \). Now assume that the result is true for \( n \) and consider the case for \( n + 1 \):

\[
s_{n+1} = 1 + \frac{s_n}{2}
\]

\[
= 1 + 1 - \frac{(2 - t)2^{1-n}}{2}
\]

\[
= 2 - (2 - t)2^{1-(n+1)}.
\]

and thus \( s_{n+1} = f(n + 1) \). Hence by mathematical induction, \( s_n = f(n) \) for all \( n \in \mathbb{N} \).

Now, by Theorem 9.7(b), we know that \( a^n \to 0 \) if \(|a| < 1\). Hence, by using Theorems 9.2 and 9.3 about the scaling and addition of sequences we know that \( s_n \to 2 - (2 - t) \cdot 0 = 2 \).