Math 104: Homework 7 solutions

1. (a) The derivative of $f(x) = \sqrt{x}$ is

$$f'(x) = \frac{1}{2\sqrt{x}}$$

which is unbounded as $x \to 0$. Since $f(x)$ is continuous on $[0, 1]$, it is uniformly continuous on this interval by Theorem 19.2. Hence for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in [0, 1]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Since this property is still satisfied if $x$ and $y$ are chosen from $(0, 1]$, then $f(x)$ is uniformly continuous on $(0, 1]$ also.

(b) Choose a $\varepsilon > 0$, and consider $x, y \in [1, \infty)$. Then

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}|$$

$$= \left| \left(\sqrt{x} - \sqrt{y}\right) \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} \right|$$

$$= \frac{|x - y|}{\sqrt{x} + \sqrt{y}}$$

$$\leq \frac{|x - y|}{2}$$

where the final line makes use of the inequality $\sqrt{x} \geq 1$ for $x \geq 1$. Hence for $|x - y| < \delta$ where $\delta = 2\varepsilon$, then

$$|f(x) - f(y)| < \varepsilon$$

and thus $f$ is uniformly continuous on $[1, \infty)$.

2. (a) Choose $\varepsilon > 0$. Then since $g$ is uniformly continuous, there exists a $\delta > 0$, such that for all $a, b \in \mathbb{R}$ with $|a - b| < \delta$,

$$|g(a) - g(b)| < \varepsilon.$$

Similarly, since $f$ is uniformly continuous, there exists $\kappa$ such that for all $x, y \in S$ with $|x - y| < \kappa$,

$$|f(x) - f(y)| < \delta.$$  

Hence, by equating $a = f(x)$ and $b = g(y)$, it can be seen that for all $|x - y| < \kappa$, where $x, y \in S$,

$$|g(f(x)) - g(f(y))| < \varepsilon$$

and thus $g \circ f$ is uniformly continuous.
(b) Choose $\epsilon > 0$. Then there exists $\delta_1 > 0$ such that for all $x, y \in S$ where $|x - y| < \delta_1$, then

$$|f(x) - f(y)| < \frac{\epsilon}{2}$$

and there exists $\delta_2 > 0$ such that for all $x, y \in S$ where $|x - y| < \delta_2$, then

$$|g(x) - g(y)| < \frac{\epsilon}{2}.$$ 

Now consider any $x, y \in S$ satisfying $|x - y| < \delta$, where $\delta = \min\{\delta_1, \delta_2\}$. Then

$$|(f + g)(x) - (f + g)(y)| = |(f(x) - f(y)) - (g(x) - g(y))|$$

$$\leq |f(x) - f(y)| + |g(x) - g(y)|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$ 

Hence $f + g$ is uniformly continuous on $S$.

(c) Consider $f(x) = x$ on $\mathbb{R}$. Then for any $\epsilon > 0$, it can be seen that for any $x, y \in \mathbb{R}$ satisfying $|x - y| < \delta$ where $\delta = \epsilon$, then $|f(x) - f(y)| < \epsilon$. Hence $f$ is uniformly continuous on $\mathbb{R}$.

Now consider $f(x) = x$ and $g(x) = x$. Then the multiplication is $h(x) = f(x) \cdot g(x) = x^2$. To show that $h$ is not continuous, pick $\epsilon = 1$, and consider any $\delta > 0$. If $x = \delta^{-1} + \frac{\delta}{2}$ and $y = \delta^{-1}$, then $|x - y| = \frac{\delta}{2}$ but

$$|h(x) - h(y)| = \left|\left(\frac{1}{\delta} + \frac{\delta}{2}\right)^2 - \frac{1}{\delta^2}\right| = 1 + \frac{\delta^2}{4} > 1$$

Hence there does not exist a $\delta > 0$ such that for all $|x - y| < \delta$, $|h(x) - h(y)| < 1$, so $h$ is not continuous on $\mathbb{R}$.

3. (a) Figure 1 shows a graph of the function $f(x) = (x - 1)^{-1}(x - 2)^{-2}$.

(b) Consider a sequence $(a_n)$ with terms in $(2, 3)$ which converges to 2. Then any term can be written at $2 + \lambda$ for some $\lambda \in (0, 1)$, and

$$f(2 + \lambda) = \frac{1}{(1 + \lambda)^2} > \frac{1}{2\lambda^2}.$$ 

Consider any $M > 0$. Then there exists an $N \in \mathbb{N}$ such that $a_n < 2 + 1/\sqrt{2M}$ for all $n > N$. Then $f(a_n) > M$ for all $n > N$, and thus $\lim_{x \to 2^+} f(x) = \infty$. Similar arguments show that

$$\lim_{x \to 2^-} f(x) = \infty,$$

$$\lim_{x \to 1^+} f(x) = \infty,$$

$$\lim_{x \to 1^-} f(x) = -\infty.$$
Figure 1: A graph of the function $f(x) = (x - 1)^{-1}(x - 2)^{-2}$.

(c) By Theorem 20.10, a limit at a point is well defined if and only if the positive and negative limits are equal. Hence $\lim_{x \to 2} f(x) = \infty$, and $\lim_{x \to 1} f(x)$ is undefined.

4. (a) Suppose that $f_1(x) \leq f_2(x)$ for all $x \in (a, b)$, but $L_1 > L_2$. Then $L_1 = L_2 + \Delta$ for some $\Delta > 0$. Now consider the sequence $c_n = a + (b - a)/(2n)$ which converges to $a$. Since $\lim_{x \to a^+} f_1(x) = L_1$, there exists an $N_1$ such that $n > N_1$ implies

$$|f_1(c_n) - L_1| < \frac{\Delta}{2}$$

and hence $f_1(c_n) - L_1 > -\Delta/2$, so that $f_1(c_n) > L_1 - \Delta/2$. Similarly there exists an $N_2$ such that $n > N_2$ implies

$$|f_2(c_n) - L_2| < \frac{\Delta}{2}$$

and hence $f_2(c_n) - L_2 < \Delta/2$, so that $f_2(c_n) < L_2 + \Delta/2$. But since $\Delta = L_1 - L_2$, then $L_1 - \Delta/2 = L_2 + \Delta/2$, and hence $f_1(c_n) < f_2(c_n)$ which is a contradiction. Hence $L_1 \leq L_2$.

(b) Consider $f_1(x) = 0$, and $f_2(x) = x$. Then for all $x \in (0, 1)$, $f_1(x) < f_2(x)$. However $\lim_{x \to 0^+} f_1(x) = 0$ and $\lim_{x \to 0^+} f_2(x) = 0$, so $L_1 = L_2$. 


5. (a) At \( x = 1 \), the series becomes \( \sum a_n \). In order for this series to converge, then \( \lim a_n = 0 \). However, if the sequence \( (a_n) \) has infinitely many non-zero integers, then there does not exist an \( N \) such that \( n > N \) implies \( |a_n - 0| < 1/2 \). Hence the series does not converge at \( x = 1 \), so the radius of convergence must be less than or equal to 1.

(b) Suppose that \( \lim \sup |a_n| = a > 0 \). Then there exist infinitely many terms \( a_{n_k} \) such that \( |a_{n_k}| > a/2 \). Now consider the sequence with terms \( |a_n|^{1/n} \). This has a subsequence \( |a_{n_k}| \), which satisfies

\[
|a_{n_k}|^{1/n_k} > \left( \frac{a}{2} \right)^{1/n_k}
\]

and as \( n_k \to \infty \), \( |a_{n_k}|^{1/n_k} \to 1 \). Hence, \( \lim \sup |a_n|^{1/n} \geq 1 \).

6. (a) For a fixed value of \( x \in [0, \infty) \),

\[
\lim_{n \to \infty} \frac{x}{n} = x \cdot \lim_{n \to \infty} \frac{1}{n} = 0.
\]

and hence the sequence of functions converges pointwise to \( f(x) = 0 \).

(b) Choose \( \epsilon > 0 \). Then let \( N = \epsilon^{-1} \). If \( n > N \), then for all \( x \in [0, 1] \),

\[
|f_n(x) - f(x)| = \left| \frac{x}{n} \right| < \frac{1}{n} < \epsilon.
\]

and hence \( f_n \to f \) uniformly on \([0, 1] \).

(c) Pick \( \epsilon = 1 \). To prove that \( f_n \) does not tend to \( f \) uniformly on \([0, \infty) \), it must be shown that there does not exist an \( N \) such that \( n > N \) implies \( |f_n(x) - f(x)| < 1 \) for all \( x \in [0, \infty) \). However, for any \( n \), if \( x = n \), then

\[
|f_n(x) - f(x)| = \left| \frac{n}{n} - 0 \right| = 1.
\]

Hence for all \( n \) there exists an \( x \) such that \( |f_n(x) - f(x)| \geq 1 \), so \( f_n \) does not converge uniformly to \( f \).

7. (a) For \( x = 0 \),

\[
\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} 0 = 0.
\]

For a fixed value of \( x \in (0, \infty) \),

\[
\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{nx}{1 + nx^2} = \lim_{n \to \infty} \frac{x}{n + 1} = \frac{x}{x^2} = \frac{1}{x}.
\]

Hence on the interval \([0, \infty) \), \( f_n \) converges pointwise to

\[
f(x) = \begin{cases} 
0 & \text{if } x = 0, \\
\frac{1}{x} & \text{if } x > 0.
\end{cases}
\]
Figure 2: Graphs of the function \( f_n(x) = \frac{nx}{1 + nx^2} \) for several values of \( n \), as well as its pointwise limit \( f(x) = x^{-2} \).

(b) To show that \( f_n \) does not converge to \( f \) uniformly on \([0, 1]\), consider \( x = \frac{1}{n} \) for \( f_n \):

\[
|f_n(x) - f(x)| = \left| \frac{1}{1 + \frac{1}{n}} - n \right| = \left| n - \frac{n}{n+1} \right|
\]

The fraction \( n/(n+1) \) is smaller than 1 for all \( n \in \mathbb{N} \). Hence for \( n \geq 2 \), and \( x = 1/n \),

\[
|f_n(x) - f(x)| > 1.
\]

Hence there does not exist an \( N \in \mathbb{N} \) such that \( n > N \) implies \( |f_n(x) - f(x)| < 1 \) for all \( x \in [0,1] \).

(c) For \( x \in [1, \infty) \),

\[
|f_n(x) - f(x)| = \left| \frac{nx}{1 + nx^2} - \frac{1}{x} \right| = \left| \frac{1}{(1 + nx^2)x} \right| < \frac{1}{n}.
\]

Thus for any \( \epsilon > 0 \), if \( N = \epsilon^{-1} \), then \( n > N \) implies that \( |f_n(x) - f(x)| < \epsilon \) for all \( x \in [1, \infty) \), and hence \( f_n \to f \) uniformly on this interval.
8. Suppose that $f(I)$ is open for any open interval $I$, but that $f$ is not monotonic. Then there would exist some interval $[a, b]$ over which it is non-monotonic.

Suppose $f(a) = f(b)$. Then if it is non-monotonic, it is non-constant so there exists an $x \in (a, b)$ such that $f(x) \neq f(a)$. Consider $f([a, b])$: by Corollary 18.3, the set must be an interval, and by Theorem 18.1 it must be bounded and attain its bounds, thus being some closed interval $[c, d]$. If $f(x) > f(a)$, then $d > f(a)$, in which case $f((a, b))$ must still contain $d$, and thus this set is not open since $d$ is non an interior point. If $f(x) < f(b)$, then $c < f(a)$, in which case $f((a, b))$ must still contain $c$, and thus this set is not open. Either possibility leads to a contradiction.

Now suppose that $f(a) < f(b)$. Then, by the argument above, $f([a, b])$ must be a closed interval $[c, d]$. If $d > f(b)$, then $d$ is still in $f((a, b))$, and thus this set is not open. Hence, since $d \geq f(b)$, then the only remaining possibility is $d = f(b)$, and hence $f(x) \leq f(b)$ for all $x \in [a, b]$. Similarly, it can be shown that $c = f(a)$, and hence $f(a) \leq f(x)$ for all $x \in [a, b]$.

Now consider the interval $[a, x]$. If $f(a) = f(x)$, this leads to a contradiction following the same argument above. Hence $f(a) < f(x)$, and thus, by applying the argument above, $f(y) \leq f(x)$ for all $y \in [a, b]$.

Since $x$ and $y$ are chosen arbitrarily, it has been shown that for all $x, y \in [a, b]$ where $x \leq y$, then $f(x) \leq f(y)$, showing that the function non-decreasing, and hence monotonic. If $f(a) > f(b)$, the above arguments can be applied to $-f$, to show that $f$ is non-increasing, and hence monotonic also. Either possibility violates the original assumption the $f$ is non-monotonic.

All possibilities lead to a contradiction, so the original assumption must be false. Hence if $f(I)$ is open for any open interval $I$, then $f$ is monotonic.