Math 126: Homework 13

1. Consider the Cauchy problem
\[ \frac{2xy}{1+y^2} u_x + u_y = -\frac{uy}{4}, \]
for the function \( u(x, y) \) with initial data \( u(x, 0) = h(x) = \sin x \).

(a) Calculate the characteristics and plot their trajectories in the \( xy \)-plane.
(b) Find \( u \) for all \( x, y \in \mathbb{R} \).
(c) Verify that \( u \) is a solution by directly substituting it into the partial differential equation.
(d) Use a computer plotting program to plot \( u \) in the domain \(-25 < x < 25 \) and \(-4 < y < 4 \).

2. Consider the linear PDE
\[ -yu_x + xu_y = 0 \]
for the function \( u(x, y) \) defined in \( \mathbb{R}^2 \), with initial data \( u(x, 0) = h(x) \). Calculate the characteristics, and determine a necessary and sufficient condition on the function \( h(x) \) for a solution to exist. In the case when this condition is satisfied, calculate the general solution of \( u(x, y) \).

3. Consider the problem
\[ u_x + 3x^2 u_y = 3ux^2 \]
for the function \( u(x, y) \) defined in \( \mathbb{R}^2 \), with initial data \( u(x, 0) = h(x) \).

(a) Calculate the characteristics, expressing them in the form \( X(s, t) \) and \( Y(s, t) \) where \( s \) is a coordinate along the initial data and \( t \) is a coordinate along each characteristic, with \( t = 0 \) corresponding to the initial data.
(b) Calculate
\[ J(s, t) = \begin{vmatrix} X_s(s, t) & Y_s(s, t) \\ X_t(s, t) & Y_t(s, t) \end{vmatrix}. \]
By considering \( J(s, 0) \), determine a necessary condition on \( h'(0) \) in order for \( C^1 \) solution to exist in a neighborhood of the \( x \) axis.
(c) Consider the case when \( h(x) = x^2 \). What is \( h'(0) \)? Find the solution \( u(x, y) \) and determine whether it is \( C^1 \).

4. Find two solutions to the equation
\[ u_x^2 + u_y^2 = 4u \]
in the disk \( \{x, y \in \mathbb{R} : x^2 + y^2 < 4\} \), with initial data \( u(\cos s, \sin s) = 1 \) for \( s \in \mathbb{R} \).