The Complexity of Symmetric Boolean Parity Holant Problems (Extended Abstract)

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Abstract. For certain subclasses of NP, ⊕P or #P characterized by local constraints, it is known that if there exist any problems that are not polynomial time computable within that subclass, then those problems are NP-, ⊕P- or #P-complete. Such dichotomy results have been proved for characterizations such as Constraint Satisfaction Problems, and directed and undirected Graph Homomorphism Problems, often with additional restrictions. Here we give a dichotomy result for the more expressive framework of Holant Problems. These additionally allow for the expression of matching problems, which have had pivotal roles in complexity theory. As our main result we prove the dichotomy theorem that, for the class ⊕P, every set of boolean symmetric Holant signatures of any arities that is not polynomial time computable is ⊕P-complete. The result exploits some special properties of the class ⊕P and characterizes four distinct tractable subclasses within ⊕P. It leaves open the corresponding questions for NP, #P and $\#_k$P for $k \neq 2$.

1 Introduction

The complexity class ⊕P is the class of languages L such that there is a polynomial time nondeterministic Turing machine that on input $x \in L$ has an odd number of accepting computations, and on input $x \notin L$ has an even number of accepting computations [29, 25]. It is known that ⊕P is at least as powerful as NP, since NP is reducible to ⊕P via (one-sided) randomized reduction [28]. Also, the polynomial hierarchy is reducible to ⊕P via two sided randomized reduction [27]. There exist decision problems, such as graph isomorphism, that are not known to be in P but are known to be in ⊕P [1]. The class ⊕P has also been related to other complexity classes via relativization [2]. Further, while the class ⊕P lies between NP and #P, it is known that there are several natural

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problems such as 2SAT that are $\oplus$P-complete where the corresponding existence problem is in P [31], and a range of others, including graph matchings and some coloring problems, for which the parity problem is in P but exact counting is #P-complete [33].

As with the classes NP and #P it is an open question whether $\oplus$P strictly extends P. For certain restrictions of these classes, however, dichotomy theorems are known. For NP a dichotomy theorem would state that any problem in the restricted subclass of NP is either in P or is NP-complete (or both, in the eventuality that NP equals P.) Ladner [24] proved that without any restrictions this situation does not hold: if P $\neq$ NP then there is an infinite hierarchy of intermediate problems that are not polynomial time interreducible.

The restrictions for which dichotomy theorems are known can be framed in terms of local constraints, most importantly, Constraint Satisfaction Problems (CSP) [26, 15, 3, 4, 17, 20, 14, 19], and Graph Homomorphism Problems [18, 21, 5]. Explicit dichotomy results, where available, manifest a total understanding of the class of computations in question, to within polynomial time reduction, and modulo the collapse of the class.

In this paper we consider dichotomies in a framework for characterizing local properties that is more general than those mentioned in the previous paragraph, and is called the Holant framework [9, 11]. A particular problem in this framework is characterized by a set of signatures as defined in the theory of Holographic Algorithms [32, 31]. The CSP framework can be viewed as the special case of the Holant framework in which equality relations of any arity are always available [11]. The addition of equality relations in CSP makes many sets of constraints complete that are not otherwise.

A brief description of the Holant framework is as follows. A *signature grid* $\Omega = (G, \mathcal{F}, \pi)$ is a triple, where $G = (V, E)$ is an undirected graph, $\mathcal{F}$ is a set of functions on variables from a domain D, and π labels each $v \in V$ with a function $f_v \in \mathcal{F}$. An assignment σ maps each edge $e \in E$ to an element of D and determines a value $\prod_{v \in V} f_v(\sigma \mid_{E(v)})$, where $E(v)$ denotes the incident edges of v, and $\sigma \mid_{E(v)}$ denotes the restriction of σ to $E(v)$. The counting problem on the instance Ω is the problem of computing the following sum over all possible assignments σ

$$\text{Holant}_\Omega = \sum_\sigma \prod_{v \in V} f_v(\sigma \mid_{E(v)}).$$

For example, consider the Perfect Matching problem on G. This corresponds to $D = \{0, 1\}$ and f_v the Exactly-One function at every vertex of G. Then σ corresponds to a subset of the edges, and Holant_Ω counts the number of perfect matchings in G. If we use the At-Most-One function at every vertex, then we count all (not necessarily perfect) matchings. We use the notation Holant($\mathcal{F}$) to denote the class of Holant problems where the functions f_v are chosen from the set $\mathcal{F}$. If all functions take integer values and we only need to compute the parity of the Holant value, it is called a *parity Holant problem*, and is denoted by $\oplus$Holant($\mathcal{F}$).

In this paper we consider symmetric boolean parity Holant problems, that is, in the definition of Holant_Ω, $D = \{0, 1\}$, $\mathcal{F}$ is a set of symmetric functions with variables in D and range in D, and summation is modulo two. Out main theorem is a dichotomy regarding the class $\oplus$P.

Theorem 1. *Let $\mathcal{F}$ be a set of symmetric signatures. The parity problem* $\oplus$Holant *($\mathcal{F}$) is either computable in polynomial time, or $\oplus$P-complete.*

This is the first such dichotomy for the Holant family. No dichotomy theorem is known for comparable restrictions of #P, NP or $\#_k$P for $k \neq 2$. For #P dichotomy results are known only for Holant^c problems, where Holant^c denotes that the unary constant signatures 0 and 1 are assumed to be available. The known results for Holant^c are for the symmetric case over the real numbers [11], and over the complex numbers [6], and for planar graphs in the former case [12]. For NP, Cook and Bruck [13] gave a dichotomy theorem for singleton sets of constraints of arity up to three in the general nonsymmetric case.

Our main dichotomy theorem exhibits four classes of signature sets that are polynomial time computable. The first class is composed by affine signatures, for which the Holant problem is solvable by Gaussian elimination. The second corresponds to signature sets that include perfect and partial matching gates. The third corresponds to Fibonacci signatures with the addition of the binary negation signature [0, 1, 0]. The fourth is what we call *vanishing signature sets*, which always give zero solutions modulo two. We do not have an explicit characterization of this fourth class. We show that any set of symmetric signatures that is not a subset of one of these four classes is $\oplus$P-complete.

Similar results have been obtained for the #CSP problem modulo k. In Faben's dichotomy theorem for boolean #CSP modulo k [19], the affine ones form the only positive class for general k, and for our case of $k = 2$ there is the second class of those that vanish for the simple reason that they are closed under complement. In the dichotomy of weighted boolean #CSP modulo k [22], the tractable classes have no immediate counterpart as it is of no meaning to discuss weights here in the parity setting.

Along the way to proving our main result we prove dichotomy theorems for both the planar and general case of $\oplus\text{Holant}^c$, that is for signature sets including both of the unary constants 0 and 1. We also prove a dichotomy theorem for 2-3 regular bipartite graphs with singleton signature sets, which is the simplest non-trivial setting, and previously investigated in the Holant framework [9, 10, 23, 6] for #P.

Finding analogs of our main result for NP, #P or $\#_k$P for $k \neq 2$ remain challenges for the future, as is also the same question for $\oplus$P for nonsymmetric signatures.

2 Preliminaries

The framework of Holant problems for #P is usually defined for functions mapping any $[q]^k \to \mathbb{C}$ for a finite q. Our results in this paper for $\oplus$P are for the

Boolean case $q = 2$ of functions $[2]^k \to \{0, 1\}$. We shall therefore assume throughout that $q = 2$.

A *signature grid* $\Omega = (H, \mathcal{F}, \pi)$ consists of a graph $H = (V, E)$ with each vertex labeled by a function $f_v \in \mathcal{F}$, where π is the labelling. The Holant problem on instance Ω is that of evaluating $\text{Holant}_\Omega = \sum_\sigma \prod_{v \in V} f_v(\sigma \mid_{E(v)})$, a sum over all edge assignments $\sigma : E \to \{0, 1\}$. A function f_v can be represented as a truth table, or as a tensor in $(\mathbb{C}^2)^{\otimes \deg(v)}$. We also use f^α to denote the value $f(\alpha)$, where α is a $\{0, 1\}$ string. A function $f \in \mathcal{F}$ is also called a *signature*. A symmetric function f on k Boolean variables can be expressed as $[f_0, f_1, \dots, f_k]$, where f_i is the value of f on inputs of Hamming weight i. For any $0 \le l < h \le k$, we call $[f_l, f_{l+1}, \dots, f_h]$ a *subsignature* of $[f_0, f_1, \dots, f_k]$. Note that with the help of the two unary signatures $[0, 1]$ and $[1, 0]$, any subsignature of a given signature is realizable.

Definition 1. *A signature is degenerate iff it is a tensor product of unary signatures.*

A Holant problem is parameterized by a set of signatures.

Definition 2. *Given a set of signatures $\mathcal{F}$, we define the following counting problem as* $\text{Holant}(\mathcal{F})$*:*

Input: A signature grid $\Omega = (G, \mathcal{F}, \pi)$;

Output: Holant_Ω.

The following family Holant^c of Holant problems is important [11, 6, 12]. This is the class of all Holant Problems (on boolean variables) where one can set any particular edge (variable) to 0 or 1 in an input to the graph, or in other words, where the unary constant functions 0 and 1 are always available for use.

Definition 3. *Given a set of signatures $\mathcal{F}$,* $\text{Holant}^c(\mathcal{F})$ *denotes* $\text{Holant}(\mathcal{F} \cup \{[1, 0], [0, 1]\})$.

In this paper, we consider the parity version of Holant problems $\oplus\text{Holant}(\mathcal{F})$, where each signature in $\mathcal{F}$ takes values from $\mathbb{Z}_2 = \{0, 1\}$. We also define $\oplus\text{Holant}^c$ problems analogously. Planar (parity) Holant problems are (parity) Holant problems on planar graphs.

To introduce the idea of holographic reductions, it is convenient first to consider bipartite graphs. We note that for any general graph we can make it bipartite by adding an additional vertex on each edge, and giving each new vertex the EQUALITY function $=_2$ on 2 inputs.

We use $\text{Holant}(\mathscr{G}|\mathscr{R})$ to denote all counting problems, expressed as Holant problems on bipartite graphs $H = (U, V, E)$, where each signature for a vertex in U or V is from $\mathscr{G}$ or $\mathscr{R}$, respectively. An input instance for the bipartite Holant problem is a bipartite signature grid and is denoted as $\Omega = (H, \mathscr{G}|\mathscr{R}, \pi)$. Signatures in $\mathscr{G}$ are denoted by column vectors (or contravariant tensors); signatures in $\mathscr{R}$ are denoted by row vectors (or covariant tensors) [16].

One can perform (contravariant and covariant) tensor transformations on the signatures. We define a simple version of holographic reductions that are invertible. Suppose Holant($\mathscr{G}|\mathscr{R}$) and Holant($\mathscr{G}'|\mathscr{R}'$) are two Holant problems defined for the same family of graphs, and $T \in \mathbf{GL}_2(\mathbb{C})$ is a basis transformation. We say that there is an (invertible) holographic reduction from Holant($\mathscr{G}|\mathscr{R}$) to Holant($\mathscr{G}'|\mathscr{R}'$), if the *contravariant* transformation $G' = T^{\otimes g}G$ and the *covariant* transformation $R = R'T^{\otimes r}$ map $G \in \mathscr{G}$ to $G' \in \mathscr{G}'$ and $R \in \mathscr{R}$ to $R' \in \mathscr{R}'$, and vice versa, where G and R have arity g and r respectively. (Notice the reversal of directions when the transformation $T^{\otimes n}$ is applied. This is the meaning of *contravariance* and *covariance*.) Suppose there is a holographic reduction from $\#\mathscr{G}|\mathscr{R}$ to $\#\mathscr{G}'|\mathscr{R}'$ mapping signature grid Ω to Ω', then $\text{Holant}_\Omega = \text{Holant}_{\Omega'}$.

In particular, for invertible holographic reductions from Holant($\mathscr{G}|\mathscr{R}$) to Holant($\mathscr{G}'|\mathscr{R}'$), one problem is in P iff the other one is in P, and similarly one problem is #P-hard ($\oplus$P-hard) iff the other one is also #P-hard ($\oplus$P-hard).

In the study of Holant problems, we will often move between bipartite and non-bipartite settings. When this does not cause confusion, we do not distinguish between signatures that are column vectors (or contravariant tensors) and row vectors (or covariant tensors). Whenever we write a transformation as $T^{\otimes n}F$ or $T\mathscr{F}$, we view the signatures as column vectors (or contravariant tensors); whenever we write a transformation as $FT^{\otimes n}$ or $\mathscr{F}T$, we view the signatures as row vectors (or covariant tensors).

All signatures we consider are in the boolean domain. If we flip the 0 and 1 in the domain, a symmetric signature will be changed into its reverse, and the Holant values are the same. That is, the complexity of Holant problems for a set of signatures is the same as the complexity of Holant problems for the set composed by those signatures reversed. In this paper this operation will be performed repeatedly.

3 Tractable Families

We shall identify three tractable families for $\oplus\text{Holant}^c$ problems. The first family, *Affine Signatures*, is adopted directly from the corresponding family for #CSP, where it is the sole tractable class [15, 14]. The second family we derive from the *Fibonacci Signatures*. For general counting problems, we also have a tractable family of Fibonacci signatures, but for parity problems, as we shall show, the family remains tractable even with the addition of the inversion signature $[0, 1, 0]$. This addition for general counting problems would give rise to #P-hardness. The third tractable family, *Matchgate Signatures*, is special to parity problems.

3.1 Affine Signatures

Definition 4. *A signature is affine iff its support is an affine space. We denote the set of all affine signatures by* $\mathscr{A}$.

By definition, an affine signature can be viewed as a constraint defined by a set of linear equations. Viewing the edges as variables in $\mathbb{Z}_2$, every assignment which contributes 1 in the summation corresponds to a solution which satisfies all the linear equations. Then the Holant value is exactly the number of solutions of the linear system, which can be computed in polynomial time.

Theorem 2. *If $\mathcal{F} \subseteq \mathscr{A}$, $\oplus\text{Holant}^c(\mathcal{F})$ is polynomial time computable.*

3.2 Fibonacci Signatures and $[0, 1, 0]$

Definition 5. *A symmetric signature $[f_0, f_1, \ldots, f_n]$ is called a Fibonacci signature iff for $1 \leq k \leq n-2$, it is the case that $x_k + x_{k+1} = x_{k+2}$. We denote the set of all Fibonacci signatures by $\mathscr{F}$.*

The family of Fibonacci signatures was introduced in [9] to characterize a new family of holographic algorithms. It has played an important role in some previous dichotomy theorems [9, 11]. The Holant of a grid composed of Fibonacci signatures can be computed in polynomial time [9]. Its parity version is therefore also tractable. But here we shall show that the tractability still holds even if we extend the set to contain the signature $[0, 1, 0]$, which is not a Fibonacci signature. This proof of tractability is based on the properties of Fibonacci signatures and a new observation on $[0, 1, 0]$ as a parity signature.

Since we only care about the parity of the solutions, $[0, 1, 0]$ can be replaced by the unsymmetrical signature $(0, 1, -1, 0)$ in $\mathbb{R}$. (Note that here $(0, 1, -1, 0)$ is not a symmetric signature. It is in fact in the vector form, rather than the abbreviated form of symmetric signatures.) This $(0, 1, -1, 0)$ is a so-called *2-realizable signature*, which is invariant up to a constant under holographic transformations [31, 7, 8]. Polynomial time computability follows from the facts that every Fibonacci signature can be transformed into a form similar to equality signatures while leaving invariant the signature $(0, 1, -1, 0)$.

Theorem 3. *If $\mathcal{F} \subseteq \mathscr{F} \cup \{[0, 1, 0]\}$, $\oplus\text{Holant}^c(\mathcal{F})$ is polynomial time computable.*

3.3 Matchgate Signatures

Definition 6. *A signature is called a matchgate signature iff it can be realized by a gadget, where each signature used in the gadget is a perfect matching signature ($[0, 1, 0, 0, \ldots, 0]$) or a partial matching signature ($[1, 1, 0, 0, \ldots, 0]$). We denote the set of all matchgate signatures by $\mathscr{M}$.*

Matchgates were introduced to simulate classically certain subclasses of quantum computations [30] and to be the basis of a class of holographic algorithms [32]. We remark that the notion of matchgate we use here is in its most general sense: the graph can be either planar or non-planar and for each node we can insist or not on whether it has to be saturated by a matching edge.

As $\mathcal{F} \subseteq \mathscr{M}$ and we also have $[1, 0], [0, 1] \in \mathscr{M}$, the problem of $\oplus\text{Holant}^c(\mathcal{F})$ is essentially that of computing the parity of the number of matchings in a graph where some specified nodes must be saturated while the remainder need not be. We show that the parity of the number of matchings equals the Pfaffian of a certain matrix of even arity in $\mathbb{Z}_2$. If the parity of the perfect matchings only is needed then such a result is immediate. What we show is that it is true also in the more general case.

Theorem 4. *If* $\mathcal{F} \subseteq \mathscr{M}$*,* $\oplus\text{Holant}^c(\mathcal{F})$ *is polynomial time computable.*

4 Hardness results and dichotomy for $\oplus\text{Holant}^c$

In this section, we prove several hardness results. These results, together with the tractable results in previous section, lead to the dichotomy theorem for $\oplus\text{Holant}^c$.

4.1 An Initial Hard Problem

The following hardness result from [31] is the starting point for all the hardness results in this paper.

Theorem 5. *⊕Pl-Rtw-Mon-3CNF is ⊕P-complete. In the Holant language, Planar* $\oplus\text{Holant}([0,1,1,1])$ *is ⊕P-complete.*

Remark: All the hardness results in this paper for $\oplus\text{Holant}^c$, but not for $\oplus\text{Holant}$, will hold even if we restrict the input to planar graphs. This is because the above starting point is true for planar graphs, and all the gadgets used in those reductions are also planar. In the following, for brevity, we will not explicitly refer to this.

This $\oplus\text{Holant}([0,1,1,1])$ can also be viewed as $\oplus\text{Holant}([1,0,1]|[0,1,1,1])$. Under the holographic transformation $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, the Holant value of $\oplus\text{Holant}\ ([1,0,1]\ |[0,1,1,1])$ is the same as that of $\oplus\text{Holant}\ ([1,1,0]|[1,0,0,1])$. This gives the following hardness result for vertex covers:

Corollary 1. $\oplus\text{Holant}([0,1,1]|[1,0,0,1])$*, and* $\oplus\text{Holant}([1,1,0]|[1,0,0,1])$ *are ⊕P-complete.*

Corollary 2. $\oplus\text{Holant}^c([0,1,1],[1,0,\ldots,0,1])$ *is ⊕P-complete, as long as the number of 0s is at least 2.*

4.2 More Hardness Results and the Dichotomy

We establish some further hardness results for other signatures. These results will be used to obtain subsequent hardness results for certain longer signatures and sets of signatures.

Lemma 1. $\oplus\text{Holant}^c([0,1,0,1,0],[0,1,1,0])$ *is ⊕P-complete.*

The proof of this lemma utilizes some novel gadget and holographic transformation. It can be further generalized because of certain realizability properties of matchgates and Fibonacci signatures.

Corollary 3. *If $\mathcal{F}$ contains a non-degenerate symmetric signature in $\mathscr{M}$ and a non-degenerate Fibonacci signature, both of which have arity at least 3, then* $\oplus\text{Holant}^c(\mathcal{F})$ *is* $\oplus$*P-complete.*

This result implies that simultaneous occurrences of matchgates and Fibonacci signatures lead to $\oplus$P-completeness. Similarly, we have the following lemma, which shows that the simultaneous occurrences of matching signatures and equality signatures also lead to $\oplus$P-completeness.

Lemma 2. *The parity problems* $\oplus\text{Holant}^c$ $([0,0,1,0], [1,0,0,\ldots,0,1])$, $\oplus\text{Holant}^c$ $([0,1,0,0], [1,0,0,\ldots,0,1])$, $\oplus\text{Holant}^c$ $([0,0,1,1], [1,0,0,\ldots,0,1])$ *and* $\oplus\text{Holant}^c$ $([1,1,0,0], [1,0,0,\ldots,0,1])$ *are all* $\oplus$*P-complete if the arity of the equality signature is at least 3.*

This lemma entails the following direct corollary for signatures that contain both equality and matching signatures as subsignatures.

Corollary 4. $\oplus\text{Holant}^c([1,0,\ldots,0,1,0])$ *and* $\oplus\text{Holant}^c([1,0,\ldots,0,1,1])$ *are* $\oplus$*P-complete, as long as the number of 0s is at least 2.*

There are still two special patterns of signatures that we need to take care of.

Lemma 3. $\oplus\text{Holant}^c([0,0,1,0,0])$ *and* $\oplus\text{Holant}^c([0,0,1,0,1])$ *are* $\oplus$*P-complete.*

Based on the algorithms in Section 3 and the hardness results above, we show a dichotomy theorem for parity Holant^c problems. The proof is basically a case-by-case study based on the number of consecutive 0s or 1s.

Theorem 6. *If* $\mathcal{F} \subseteq \mathscr{A}$, $\mathcal{F} \subseteq \mathscr{M}$ *or* $\mathcal{F} \subseteq \mathscr{F} \cup \{[0,1,0]\}$ *then the parity problem* $\oplus\text{Holant}^c(\mathcal{F})$ *is computable in polynomial time. Otherwise it is* $\oplus$*P-complete. The same statement also holds for planar graphs.*

5 Vanishing Signature Sets

In the remaining two sections we extend our results to obtain the dichotomy result for $\oplus$Holant without any assumptions. In order to formulate the dichotomy we shall need a fourth family of tractable signature sets, which we call *Vanishing Signature Sets*.

Definition 7. *A set of signatures $\mathcal{F}$ is called vanishing iff the value of* $\oplus\text{Holant}_\Omega$ $(\mathcal{F})$ *is zero for every Ω. We denote the class of all vanishing signature sets by* $\mathscr{O}$.

First we show some general properties of vanishing signature sets. For two signatures f and g of the same arity, $f+g$ denotes the bitwise addition in $\mathbb{Z}_2$, i.e. $[f_0+g_0, f_1+g_1, \ldots]$.

Lemma 4. *Let $\mathcal{F}$ be a vanishing signature set. If a signature f can be realized by a gadget using signatures in $\mathcal{F}$, then $\mathcal{F} \cup \{f\} \in \mathscr{O}$. If g_0 and g_1 are two signatures in $\mathcal{F}$ with the same arity, then $\mathcal{F} \cup \{g_0 + g_1\} \in \mathscr{O}$.*

There are several classes of vanishing signatures, e.g. complement invariant signatures and matchgate-based vanishing signatures. Here we introduce a concept called *self-vanishable signatures* which plays an important role in the proof of the general dichotomy. First, we introduce an extended version of the inner product for two signatures of not necessarily the same arity.

Definition 8. *Let f and g be two signatures with arities n and m ($n \geq m$) respectively. Their inner product $h = \langle f, g \rangle$ is a signature with arity $n - m$ defined as follows:*

$$h^\alpha = \sum_{\beta \in \{0,1\}^m} f^{\beta,\alpha} g^\beta,$$

where $\alpha \in \{0,1\}^{n-m}$.

If f is symmetric, the final $h = \langle f, g \rangle$ is also symmetric. If both f and g are symmetric, their inner product $h = [h_0, h_1, \ldots, h_{n-m}]$ has the following form: $h_i = \sum_{j=0}^{m} \binom{m}{j} f_{j+i} g_j$ for $0 \leq i \leq n - m$.

Definition 9. *A signature f is called self-vanishable of degree k iff $\langle f, [1,1]^{\otimes k} \rangle = \mathbf{0}$ and $\langle f, [1,1]^{\otimes k-1} \rangle \neq \mathbf{0}$. We denote this by $v(f) = k$. If such a k does not exist, the signature f is not self-vanishable.*

We note that for the trivial signature $\mathbf{0}$, we have $v(\mathbf{0}) = 0$. Also, $f = [1,1]$ is self-vanishable with $v(f) = 1$ since $\langle [1,1], [1,1] \rangle = 0$.

For a symmetric signature $f = [f_0, f_1, \ldots, f_n]$, we call f_0 the first entry of f and $f_0, f_1, ..., f_{k-1}$ the first k entries of f. It follows from the definition that for a symmetric signature $f = [f_0, f_1, \ldots, f_n]$, we have

$$\langle f, [1,1] \rangle = [f_0 + f_1, f_1 + f_2, \ldots, f_{n-1} + f_n].$$

Hence the only symmetric signature of arity n with $v(f) = 1$ is $[1,1]^{\otimes n}$. There are two symmetric signatures of arity $n \geq 3$ with $v(f) = 2$, which are the parity signatures $[1,0,1,0,\ldots,0/1]$ and $[0,1,0,1,\ldots,0/1]$. Inductively, we have the following lemma:

Lemma 5. *For any $k \geq 2$, there are 2^{k-1} symmetric signatures of arity $n \geq k$ with $v(f) = k$, whose first $k-1$ entries are arbitrary and the remaining entries are determined by them.*

To be self-vanishable is a necessary condition for a signature to be a member of a vanishing signature set. This lemma also explains the intuition for why we define this notion of self-vanishable and why we define it in this way. The proof is a direct construction of a grid with Holant value 1.

Lemma 6. *If $\mathcal{F}$ contains a signature f which is not self-vanishable then $\mathcal{F}$ is not a vanishing set.*

However, it is not sufficient for a signature to be self-vanishable for it to form a vanishing set. One condition that is sufficient, called strong self-vanishable, is defined below. There exist some weak self-vanishable signatures that do not form vanishing sets, e.g. $\{[1, 0, 0, 0, 1, 0]\}$.

Definition 10. *Let f be self-vanishable of degree $k \geq 0$ with arity n. It is called strong self-vanishable if $k \leq \lfloor \frac{n}{2} \rfloor + 1$ and weak self-vanishable if $\lfloor \frac{n}{2} \rfloor + 2 \leq k \leq n$.*

Theorem 7. *Let $\mathcal{F}$ be a set of symmetric strong self-vanishable signatures. Then $\mathcal{F}$ is a vanishing set, i.e. $\mathcal{F} \in \mathscr{O}$.*

As a final remark we note that the family $\mathscr{O}$ of vanishing signature sets has the following difference from the previous tractable families $\mathscr{A}$, $\mathscr{M}$ and $\mathscr{F} \cup \{[0, 1, 0]\}$. The union of two sets in $\mathscr{O}$ is not necessarily in $\mathscr{O}$.

6 Dichotomy for The Whole Holant Family

In this final section, we prove our main theorem, the dichotomy for all parity Holant problems with symmetric signatures, without assuming any freely available signatures. This improves on our dichotomy theorem for parity Holant^c problems given in Section 4, which we use, however, as our starting point. The main idea is to construct gadgets for the two signatures $[0, 1]$ and $[1, 0]$. We will first show that realizing either one of these is enough. Where one of these unary signatures is realizable, we reduce the Holant problem to the corresponding Holant^c problem and apply the Holant^c dichotomy result. However, for some signature sets it is impossible to realize $[0, 1]$ or $[1, 0]$. We show that those signature sets must be vanishing, in the sense defined in the previous section.

First we show that it is enough to realize just one of $[0, 1]$ or $[1, 0]$. We remark that the gadgets used in the proof are not all planar, and hence the dichotomy for planar graphs does not follow.

Lemma 7. *Let $\mathcal{F}$ be a set of symmetric signatures. If $\mathcal{F} \subseteq \mathscr{A}$, $\mathcal{F} \subseteq \mathscr{M}$, or $\mathcal{F} \subseteq \mathscr{F} \cup \{[0, 1, 0]\}$ then the parity problems $\oplus$Holant$(\mathcal{F} \cup \{[1, 0]\})$, $\oplus$Holant$(\mathcal{F} \cup \{[1, 0, 0]\})$, $\oplus$Holant$(\mathcal{F} \cup \{[0, 1]\})$ and $\oplus$Holant$(\mathcal{F} \cup \{[0, 0, 1]\})$ are computable in polynomial time. Otherwise these parity problems are $\oplus$P-complete.*

The proof of the first part of Lemma 7 is a case-by-case study which shows that we can always realize $[0, 1]$ via some signature in $\mathcal{F}$ and $[1, 0]$. The second part is shown using reductions from $\oplus$Holant$(\mathcal{F} \cup \{[1, 0]\})$ to $\oplus$Holant$(\mathcal{F} \cup \{[1, 0, 0]\})$. We duplicate an instance of $\oplus$Holant$(\mathcal{F} \cup \{[1, 0]\})$, and replace every two corresponding occurrences of $[1, 0]$ with a binary signature $[1, 0, 0]$. The Holant value remains the same due to properties of $\mathbb{Z}_2$.

An important fact is that, as long as the signature set $\mathcal{F}$ is not vanishing, one of the above unary or binary signatures is realizable. In particular, we can always construct such a signature from a not self-vanishable signature, or realize either $[0,1]$ or $[1,0]$ from a weak self-vanishable signature. This gives our main theorem.

Theorem 8. *Let $\mathcal{F}$ be a set of symmetric signatures. If $\mathcal{F} \subseteq \mathscr{A}$, $\mathcal{F} \subseteq \mathscr{M}$, $\mathcal{F} \subseteq \mathscr{F} \cup \{[0,1,0]\}$, or $\mathcal{F} \in \mathscr{O}$ then the parity problem $\oplus$Holant$(\mathcal{F})$ is computable in polynomial time. Otherwise it is $\oplus$P-complete.*

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